Modèle de Maxwell :

Solution homogène : grâce à l'article de Brey, on a directement la solution homogène qui est celle du gaz de sphères dures, à la différence de la densité (et evtl. de la température à l'ordre non homogène) qui décroit :

$$f^{(0)}(\underline{v}) = \frac{n}{v_T^d} e^{-v^2/v_T^2} \frac{1}{\pi^{d/2}} \; , \; \mathcal{M}\left(\frac{v}{v_T}\right) = \frac{1}{\pi^{d/2}} e^{-v^2/v_T^2} \frac{n}{v_T^d} \; ; \; v_T = \sqrt{2/\beta m} \quad : \quad f^{(0)}(\underline{v}) = \mathcal{M}(v/v_T)$$

En particulier, l'état est caractérisé par $a_2 = 0$. Pour l'état inhomogène :

$$(\partial_t + \underline{v}\cdot\underline{\nabla}) f(\underline{r}, \underline{v}; t) = p J_a[f,f] + (1-p) J_c[f,f]$$

$$J_a[f,g] = -\sigma^{d-1} \frac{\phi v_T}{S_d} \int dv_2 \int d\hat{\sigma}\, f(\underline{r}, \underline{v_2}; t)\, g(\underline{r}, \underline{v_1}; t) \qquad ; \; S_d = \int d\hat{\sigma} = 2\pi^{d/2}/\Gamma(d/2)$$

$$= -\sigma^{d-1} \frac{\phi v_T}{S_d} g(\underline{r}, \underline{v_1}; t) \int d\hat{\sigma} \int dv_2\, f(\underline{r}, \underline{v_2}; t)$$

$$= -\sigma^{d-1} \phi\, v_T\, g(\underline{r}, \underline{v_1}; t) \int dv_2\, f(\underline{r}, \underline{v_2}; t)$$

$$J_c[f,g] = \sigma^{d-1} \frac{\phi v_T}{S_d} \int d\underline{v_2} \int d\hat{\sigma}\, (b^{-1} - 1)\, g(\underline{r}, \underline{v_1}; t) f(\underline{r}, \underline{v_2}; t)$$

Dans le modèle de Maxwell, qui n'a pas de dérivation microscopique, la section efficace est en $1/|v_{12}|$ et donc dans le taux de collision on remplace le terme restant $|v_{12}\cdot\hat{\sigma}|$ par une valeur moyenne proportionnelle à la vitesse thermique, d'où le $1/S_d$ par la moyenne sur toutes les directions. L'amplitude ϕ reste un paramètre libre, à fixer ultérieurement. Les Eqs. de bilan s'obtiennent de la m. façon, avec [Eqs. (4.28)]

$$\omega[f,g] = -\int_{\mathbb{R}^d} dv_1\, J_a[f,f] = \sigma^{d-1} \phi\, v_T \int_{\mathbb{R}^d} dv_1\, g(\underline{r}, \underline{v_1}; t) \int_{\mathbb{R}^d} dv_2\, f(\underline{r}, \underline{v_2}; t)$$

Solution de Chapman-Enskog : ordre zéro : même forme :

$$\partial_t n = -p n\, \zeta_n^{(0)},$$
$$\partial_t u_i = -p v_T\, \zeta_{u_i}^{(0)},$$
$$\partial_t T = -p T\, \zeta_T^{(0)},$$

avec :

$$\zeta_n^{(0)} = \frac{1}{n}\omega[f^{(0)}, f^{(0)}] = \frac{1}{n}\sigma^{d-1}\phi\, v_T \left[\int_{\mathbb{R}^d} dv\, f^{(0)}(\underline{r}, \underline{v}; t)\right]^2 = \frac{1}{n}\sigma^{d-1}\phi\, v_T n^2 = n\sigma^{d-1}\phi\, v_T$$

$$\zeta_{u_i}^{(0)} = \frac{1}{n v_T}\omega[f^{(0)}, v_i f^{(0)}] = \frac{1}{n v_T}\sigma^{d-1}\phi\, v_T \underbrace{\int_{\mathbb{R}^d} dv_2\, f^{(0)}(\underline{r}, \underline{v_2}, t)}_{=n} \underbrace{\int_{\mathbb{R}^d} dv_1\, f^{(0)}(\underline{r}, \underline{v_1}; t)\, v_i}_{=0} = 0$$

$$\zeta_T^{(0)} = \frac{m}{n k_B T d}\omega[f^{(0)}, v^2 f^{(0)}] - \frac{1}{n}\omega[f^{(0)}, f^{(0)}]$$

$$= \frac{m}{n k_B T d}\sigma^{d-1}\phi\, v_T \int_{\mathbb{R}^d} dv_1\, f^{(0)}(\underline{r}, \underline{v_1}; t)\, v_1^2 \underbrace{\int_{\mathbb{R}^d} dv_2\, f^{(0)}(\underline{r}, v_2; t)}_{=n} - n\sigma^{d-1}\phi\, v_T$$

$$= \underbrace{\left(\frac{m}{n k_B d}\int_{\mathbb{R}^d} dv_1\, f^{(0)}(\underline{r}, \underline{v_1}; t)\right)}_{=T} \frac{1}{T} n\sigma^{d-1}\phi\, v_T - n\sigma^{d-1}\phi\, v_T$$

$$= n\sigma^{d-1}\phi\, v_T - n\sigma^{d-1}\phi\, v_T$$
$$= 0$$

$$\phi = \frac{4}{\sqrt{2}} \frac{\pi^{(d-1)/2}}{\Gamma(d/2)} \; ; \; \nu_0 = \frac{n k T}{d+2} 8 \frac{\pi^{(d-1)/2}}{\Gamma(d/2)} \frac{\sigma^{d-1}}{\sqrt{m k T}}$$

$$\Rightarrow \zeta_n^{(0)*} = \zeta_n^{(0)}/\nu_0$$

$$= \cancel{n\sigma^{d-1}} v_T \frac{4}{\sqrt{2}} \frac{\cancel{\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)}} \frac{d+2}{\cancel{nkT}} \frac{1}{8} \frac{\cancel{\Gamma(d/2)}\sqrt{mkT}}{\cancel{\pi^{(d-1)/2}\sigma^{d-1}}}$$

$$= v_T \frac{4}{\sqrt{2}} \frac{d+2}{8} \sqrt{m}\sqrt{\beta}$$

$$= \sqrt{2/\beta m}\sqrt{m\beta}\,\frac{d+2}{2}\,\frac{1}{\sqrt{2}}$$

$$= \frac{d+2}{\cancel{4}\,2}$$

$$\zeta_n^{(0)*} = \cancel{n\sigma^{d-1}}\phi\sqrt{\frac{2}{\cancel{\beta m}}}\frac{1}{\cancel{nkT}}\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{\cancel{\sqrt{mkT}}}{\cancel{\sigma^{d-1}}}$$

$$= \phi\frac{d+2}{8}\frac{\sqrt{2}\,\Gamma(d/2)}{\pi^{(d-1)/2}}$$

$$= \frac{4}{\sqrt{2}}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}\frac{\sqrt{2}\,\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{d+2}{8}$$

$$= \frac{d+2}{2}$$

On en conclut que $v_T = \sqrt{2/\beta m}$ ne dépend pas de t pour l'état homogène, et donc la solution $\partial_t n = -p n\,\zeta_n^{(0)}$ est facile :

$$n_H(t) = \frac{n_0}{1 + p t\, \zeta_n^{(0)}(0)} \quad ; \; \zeta_n^{(0)}(0) = n\sigma^{d-1}\phi\, v_T \quad ; \; v_T = \sqrt{2/\beta_H m} \; ; \; \beta_H = k_B T_H$$

Ordre 1 : les calculs sont similaires et on obtient les Eqs. (4.46)–(4.51).

Coefficients de transport : différence : calcul de

$$\frac{1}{dT}\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v}) A_i(\underline{v}) \; ; \; \frac{1}{dn}\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v}) B_i(\underline{v})$$

$$A_i = \frac{v_i}{2}\frac{\partial}{\partial v_j}[v_j f^{(0)}] - \frac{k_B T}{m}\frac{\partial f^{(0)}}{\partial v_i}$$

$$B_i = -v_i f^{(0)} - \frac{k_B T}{m}\frac{\partial f^{(0)}}{\partial v_i}$$

En effet, l'expression de $f^{(0)}$ diffère. Par contre, pour arriver aux coefficients de transport ce sont des résultats généraux qui utilisent uniquement

$$a_2 = \frac{4}{d(d+2)}\left(\frac{\beta m}{2}\right)^2 \frac{1}{n}\int_{\mathbb{R}^d} d\underline{v}\, v^4 f^{(0)} - 1.$$

Comme dans notre cas $a_2 = 0$, il suffit de poser $a_2 = 0$ dans les Eqs. (4.56) :

$$\eta^* = \frac{\eta}{\eta_0} = \frac{1}{\nu_\eta^* - \frac{1}{2}p\,\underbrace{\zeta_T^{(0)*}}_{=0}} = \frac{1}{\nu_\eta^*}$$

$$\kappa^* = \frac{\kappa}{\kappa_0} = \frac{1}{\nu_\kappa^* - 2p\,\cancel{\zeta_T^{(0)*}}}\left[\frac{1}{2}p\,\zeta_n^{(0)*}\,\underbrace{\cancel{\mu^*}}_{=0} + \frac{d-1}{d}(2\underbrace{\cancel{a_2}}_{=0} + 1)\right] = \frac{d-1}{d}\frac{1}{\nu_\kappa^*}$$

$$\mu^* = \frac{n\mu}{T\kappa_0} = \frac{2}{2\nu_\mu^* - 3p\,\underbrace{\cancel{\zeta_T^{(0)*}}}_{=0} - 2p\,\zeta_n^{(0)*}}\left[p\,\underbrace{\cancel{\zeta_T^{(0)*}}}_{=0}\kappa^* + \frac{d-1}{d}\underbrace{\cancel{a_2}}_{=0}\right] = 0$$

Avec η_0, κ_0, ν_0 donnés par les sphères dures. C'est en calculant ν_η^* et ν_κ^* qu'on fait des approximations supplémentaires, peut-être de trop pour le modèle de Maxwell car on pourrait peut-être résoudre exactement. En particulier : $\nu_0 = n\sigma^{d-1}\frac{8}{d+2}\sqrt{2}\, v_T\, \pi^{(d-1)/2}/\Gamma(d/2)$

A calculer :

+ refaire avec coeff. transport exacts.
Permet de : – quantifier l'erreur de l'approx. de $f^{(1)}$ par le premier polynôme Sonine
– discuter les divergences modèle Maxwell ⟺ sphères dures
– impact sur les relations de dispersion
– discuter la limite p plus proche de 1, car on a des résultats exacts donc meilleurs ⇒ meilleure discussion de la validité de l'approche hydrodynamique.

$$V_\eta^* = \frac{1}{v_0}\frac{\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, J\, Z_{ij}}{\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, Z_{ij}} - p\,\frac{1}{v_0}\frac{\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\,\Omega\, Z_{ij}}{\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, Z_{ij}} = (1-p)V_\eta^{*c} + p V_\eta^{*a}$$

approx.
$$\begin{cases} V_\eta^{*a} = \dfrac{\beta^2}{(d+2)(d-1)n v_0}\displaystyle\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, L_a[\mathcal{M} D_{ij}] + V_\eta^{*a'} \\ V_\eta^{*a'} = -\dfrac{\beta^2}{(d+2)(d-1)n v_0}\displaystyle\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\,\Omega[\mathcal{M} D_{ij}] = 0 \\ V_\eta^{*c} = \dfrac{\beta^2}{(d+2)(d-1)n v_0}\displaystyle\int_{\mathbb{R}^d} dv\, D_{ij}(\underline{v})\, L_c[\mathcal{M} D_{ij}] \end{cases}$$

$$V_\kappa^* = \frac{1}{v_0}\frac{\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\, J\, A_i}{\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\, A_i} - p\,\frac{1}{v_0}\frac{\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\,\Omega\, A_i}{\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\, A_i} = (1-p)V_\kappa^{*c} + p V_\kappa^{*a}$$

approx.
$$\begin{cases} V_\kappa^{*a} = \dfrac{2m\beta^3}{d(d+2)n v_0}\displaystyle\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\, L_a[\mathcal{M} S_i] + V_\kappa^{*a'} \\ V_\kappa^{*a'} = -\dfrac{2m\beta^3}{d(d+2)n v_0}\displaystyle\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\,\Omega[\mathcal{M} S_i] = 0 \\ V_\kappa^{*c} = \dfrac{2m\beta^3}{d(d+2)n v_0}\displaystyle\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v})\, L_c[\mathcal{M} S_i] \end{cases}$$

Lemme 3.3 : $\int_{\mathbb{R}^d} d\underline{v}_1\, Y(\underline{v}_1)\, L_a[\mathcal{M}X] = \sigma^{d-1}\phi\, v_T \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)\, X(\underline{v}_2)\,[Y(\underline{v}_1)+Y(\underline{v}_2)]$

Lemme 3.4 : $\int_{\mathbb{R}^d} d\underline{v}_1\, Y(\underline{v}_1)\, L_c[\mathcal{M}X] = -\sigma^{d-1}\frac{\phi v_T}{S_d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)X(\underline{v}_2)\int d\hat\sigma\,(b-1)\,[Y(\underline{v}_1)+Y(\underline{v}_2)]$; $b\underline{v}_i = \underline{v}_i \mp (\underline{v}_{12}\cdot\hat\sigma)\hat\sigma$

<u>Calcul de V_η^{*a}</u> :

$$\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, L_a[\mathcal{M}D_{ij}] \overset{3.3.}{=} \sigma^{d-1}\phi v_T\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)\, D_{ij}(\underline{v}_2)\,[D_{ij}(\underline{v}_1)+D_{ij}(\underline{v}_2)]$$

$$\cancel{= \sigma^{d-1}\phi v_T\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2 \frac{n}{v_T^d}\frac{1}{\pi^{d/2}}e^{-v_1^2/v_T^2}\frac{n}{v_T^d}\frac{1}{\pi^{d/2}}e^{-v_2^2/v_T^2} m^2\left[(\underline{v}_1\cdot\underline{v}_2)^2 - \frac{1}{d}v_1^2v_2^2 + v_2^4 - \frac{1}{d}v_2^4\right]}$$

$$= \sigma^{d-1}\phi v_T\frac{n^2}{v_T^{2d}}\frac{m^2}{\pi^d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, e^{-v_1^2/v_T^2}e^{-v_2^2/v_T^2}\left[\cancel{\frac{1}{d}v_1^2v_2^2 - \frac{1}{d}v_1^2v_2^2} + \frac{d-1}{d}v_2^4\right]$$

$$= \sigma^{d-1}\phi v_T\frac{n^2}{\cancel{v_T^{2d}}}\frac{m^2}{\pi^d}\cancel{v_T^{2d}}\, v_T^4\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}\, c_2^4\,\frac{d-1}{d}$$

$$= \sigma^{d-1}\phi\,\frac{m^2n^2}{\cancel{\pi^d}}\, v_T^5\,\frac{d-1}{d}\,\cancel{\pi^{d/2}}\cancel{\frac{\Gamma(d/2)}{\Gamma(d/2)}}\,\cancel{\pi^{d/2}}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}$$

$$= \sigma^{d-1}\phi\, m^2n^2\, v_T^5\,\frac{d-1}{\cancel{d}}\,\frac{d+2}{2}\,\frac{\cancel{d}}{2}$$

$$\frac{\beta^2}{(d+2)(d-1)nv_0}\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, L_a[\mathcal{M}D_{ij}] = \sigma^{d-1}\phi\,\frac{\beta^2}{\cancel{(d+2)(d-1)}nv_0}\, m^2n^{\cancel{2}}v_T^5\,\frac{\cancel{(d-1)}\cancel{(d+2)}}{4} = \frac{\sigma^{d-1}\beta^2m^2nv_T^5}{4v_0}$$

$$= \frac{\cancel{\sigma^{d-1}}\cancel{\beta^2}\cancel{m^2}\cancel{n}}{4}\left(\frac{2}{\beta m}\right)^2 \cancel{v_T}\,\frac{1}{\cancel{n\sigma^{d-1}}}\,\frac{d+2}{8}\,\frac{\sqrt{2}\,\Gamma(d/2)}{\cancel{\sqrt{2}}\cancel{v_T}\,\pi^{(d-1)/2}}\,\phi$$

$$= \phi\,\frac{d+2}{8}\,\frac{\Gamma(d/2)\sqrt{2}}{\sqrt{2}\,\pi^{(d-1)/2}}$$

<u>Calcul de V_η^{*c}</u> :

$$\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, L_c[\mathcal{M}D_{ij}] \overset{3.4.}{=} -\sigma^{d-1}\frac{\phi v_T}{S_d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)\, D_{ij}(\underline{v}_2)\int d\hat\sigma\,(b-1)\,[D_{ij}(\underline{v}_1)+D_{ij}(\underline{v}_2)]$$

cf notes

$$= -\sigma^{d-1}\frac{\phi v_T}{S_d}\, m\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)\, D_{ij}(\underline{v}_2)\int d\hat\sigma\,(g\cdot\hat\sigma)\,[-g_i\sigma_j - g_j\sigma_i + 2(g\cdot\hat\sigma)\sigma_i\sigma_j] \quad ;\ \underline{g} = \underline{v}_1 - \underline{v}_2$$

On utilise :

$$\int d\hat\sigma\,(g\cdot\hat\sigma)^n\hat\sigma_i = 2\beta_{n+1}\hat g_i$$
$$\int d\hat\sigma\,(\hat g\cdot\hat\sigma)^n\hat\sigma_i\hat\sigma_j = \frac{2\beta_n}{n+d}(n\hat g_i\hat g_j + \delta_{ij})$$
$$\beta_n = \frac{1}{2}\int d\hat\sigma\,(\hat\sigma\cdot\hat g)^n = \pi^{(d-1)/2}\,\Gamma\!\left(\tfrac{n+1}{2}\right)/\Gamma\!\left(\tfrac{n+d}{2}\right)$$

Ainsi :

$$\int d\hat\sigma\,(g\cdot\hat\sigma)(-g_i\sigma_j) = -g_i\, g\int d\hat\sigma\,(\hat g\cdot\hat\sigma)\sigma_j = -g_i\, g_j\, 2\beta_2$$
$$\int d\hat\sigma\,(g\cdot\hat\sigma)(-g_j\sigma_i) = -g_j g_i\, 2\beta_2$$
$$\int d\hat\sigma\,(g\cdot\hat\sigma)\, 2(g\cdot\hat\sigma)\sigma_i\sigma_j = 2\int d\hat\sigma\,(g\cdot\hat\sigma)^2\sigma_i\sigma_j = \frac{4\beta_2}{d+2}(2g_ig_j + g^2\delta_{ij})$$

somme :
$$4\beta_2\left[-g_ig_j + \frac{2g_ig_j}{d+2} + \frac{g^2}{d+2}\delta_{ij}\right] = \frac{4\beta_2}{d+2}\left[g_ig_j(\cancel{2} - d - \cancel{2}) + g^2\delta_{ij}\right]$$

Ainsi :

$$\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\, L_c[\mathcal{M}D_{ij}] = -\sigma^{d-1}\frac{\phi v_T}{S_d}\, m\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)\, D_{ij}(\underline{v}_2)\,\frac{4\beta_2}{d+2}\Big[\underbrace{(v_1-v_2)^2}_{v_1^2+v_2^2-2(\underline{v}_1\cdot\underline{v}_2)}\delta_{ij} - d\,\underbrace{(v_1-v_2)_i(v_1-v_2)_j}_{v_{1i}v_{1j} - v_{1i}v_{2j} - v_{2i}v_{1j} + v_{2i}v_{2j}}\Big]$$

$$= -\sigma^{d-1}\frac{\phi v_T}{S_d}\, m^2\frac{4\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, f^{(0)}(\underline{v}_1)\mathcal{M}(\underline{v}_2)\left(v_{2i}v_{2j} - \frac{1}{d}v_2^2\delta_{ij}\right)\left[(v_1^2+v_2^2-2(v_1\cdot v_2))\delta_{ij} - d\,v_{1i}v_{1j} - d\,v_{1i}v_{2j} - d\,v_{2i}v_{1j} + d\,v_{2i}v_{2j}\right]$$

Raisons de symétrie dans les intégrales :

$$v_{2i}v_{2j}\, v_1^2\,\delta_{ij} = v_1^2\, v_2^2$$
$$v_{2i}v_{2j}\, v_2^2\,\delta_{ij} = v_2^2\, v_2^2 = v_2^4$$
$$v_{2i}v_{2j}\,\delta_{ij}\,(-2)\, v_{1k}v_{2k} = -2\,v_2^2\;\underline{v}_1\cdot\underline{v}_2 \longrightarrow 0 :\ \text{antisymétrique}$$

$$V_{2i}V_{2j}\left(-d\,V_{1i}V_{1j}\right) = -d\,(V_1\cdot V_2)^2 \longrightarrow -d\,\frac{1}{d}V_1^2V_2^2 = -V_1^2V_2^2$$
$$V_{2i}V_{2j}\left(-d\,V_{1i}V_{2j}\right) = -d\,V_{2i}V_{1i}V_2^2 \longrightarrow 0$$
$$V_{2i}V_{2j}\left(-d\,V_{2i}V_{1j}\right) = -d\,V_2^2V_{2j}V_{1j} \longrightarrow 0$$
$$V_{2i}V_{2j}\,d\,V_{2i}V_{2j} = d\,V_2^2V_2^2 = d\,V_2^4$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\,V_1^2\,\delta_{ij} = -V_1^2V_2^2$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\,V_2^2\,\delta_{ij} = -V_2^4$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\left[-2(V_1\cdot V_2)\right]\delta_{ij} = \frac{2}{d}d\,V_2^2(V_1\cdot V_2) = 2V_2^2(V_1\cdot V_2)\longrightarrow 0$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\left[-d\,V_{1i}V_{1j}\right] = V_2^2\,\delta_{ij}V_{1i}V_{1j} = V_1^2V_2^2$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\left[-d\,V_{1i}V_{2j}\right] = V_2^2\,V_{1i}V_{2i} = V_2^2(V_1\cdot V_2)\longrightarrow 0$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\left[-d\,V_{2i}V_{1j}\right] = V_2^2(V_1\cdot V_2)\longrightarrow 0$$
$$-\frac{1}{d}V_2^2\,\delta_{ij}\left[d\,V_{2i}V_{2j}\right] = -V_2^2\,V_{2i}V_{2i} = -V_2^4$$

Tout ensemble :

$$\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v})\,L_c[\mathcal{M}D_{ij}] = +\sigma^{d-1}\frac{\phi V_T}{S_d}m^2\frac{4\beta_2}{d+2}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{V}_1)\mathcal{M}(\underline{V}_2)\left[\cancel{V_1^2V_2^2}+\cancel{V_2^4}-\cancel{V_1^2V_2^2}+dV_2^4-\cancel{V_1^2V_2^2}-\cancel{V_2^4}+\cancel{V_1^2V_2^2}-V_2^4\right]$$

$$=+\sigma^{d-1}\frac{\phi V_T}{S_d}m^2\frac{4\beta_2}{d+2}(d-1)\underbrace{\int_{\mathbb{R}^d} d\underline{v}_1 f^{(0)}(\underline{V}_1)}_{=n}\underbrace{\int_{\mathbb{R}^d} d\underline{v}_2\,\mathcal{M}(\underline{V}_2)\,V_2^4}_{=\cancel{V_T^d}V_T^4\frac{n}{\cancel{V_T^d}}\int_{\mathbb{R}^d}dc\frac{1}{\pi^{d/2}}e^{-c^2}c^4}$$

$$=+\sigma^{d-1}\frac{\phi V_T^5}{S_d}m^2 4\beta_2\frac{d-1}{d+2}n^2\frac{1}{\cancel{\pi^{d/2}}}\cancel{\pi^{d/2}}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}$$

$$=+\sigma^{d-1}\frac{\phi V_T^5}{S_d}m^2n^2\beta_2\frac{\cancel{4}(d-1)}{\cancel{d+2}}\frac{\cancel{d+2}}{\cancel{2}}\frac{d}{\cancel{2}}$$

$$=+\sigma^{d-1}\frac{\phi V_T^5}{S_d}m^2n^2\beta_2\,d(d-1)$$

$$\frac{\beta^2}{(d+2)(d-1)nV_0}\int_{\mathbb{R}^d}d\underline{v}\,D_{ij}(\underline{v})\,L_c[\mathcal{M}D_{ij}] = +\sigma^{d-1}\frac{\phi}{S_d}\left(\frac{2}{\cancel{\beta m}}\right)^2 V_T\,\cancel{m^2}n^{\cancel{2}}\beta_2 d\cancel{(d-1)}\frac{\cancel{\beta^2}}{(d+2)\cancel{(d-1)}\cancel{n}V_0}$$

$$=+\cancel{\sigma^{d-1}}\phi\,4\,\cancel{V_T}\frac{\Gamma(d/2)}{2\pi^{d/2}}\cancel{n}\beta_2 d\frac{1}{\cancel{d+2}}\frac{1}{\cancel{\sigma^{d-1}}}\frac{\cancel{d+1}}{8}\frac{\Gamma(d/2)\,\sqrt{2}}{\cancel{n}\cancel{V_T}\pi^{(d-1)/2}} \quad ;\ \beta_2=\pi^{(d-1)/2}\frac{\Gamma(1+1/2)}{\Gamma(\frac{d+2}{2})}$$

$$=+\phi\cancel{4}\frac{\Gamma(d/2)}{2\pi^{d/2}}\cancel{\pi^{(d-1)/2}}\frac{\Gamma(1/2)1/2}{\cancel{\frac{d}{2}}\cancel{\Gamma(d/2)}}\frac{\cancel{d}}{\cancel{8}}\frac{\cancel{\Gamma(d/2)}\sqrt{2}}{\cancel{n}\cancel{\pi^{(d-1)/2}}}$$

$$=+\phi\frac{\Gamma(d/2)}{2\pi^{d/2}}\frac{1}{2}\sqrt{\pi}\frac{1}{(\sqrt{2})^{-1}}$$

$$=+\phi\frac{\Gamma(d/2)\sqrt{2}}{4\cancel{n}\pi^{(d-1)/2}}$$

Calcul de $V_1^{*\alpha}$: de l'Eq. (443) de mes notes on a :

$$\Omega[\mathcal{M}D_{ij}] = f^{(0)}\frac{2}{n}\omega[f^{(0)},\mathcal{M}D_{ij}] + \frac{\partial f^{(0)}}{\partial T}T\left[-\frac{2}{n}\omega[f^{(0)},\mathcal{M}D_{ij}] + \frac{m}{nk_BTd}\omega[f^{(0)},V^2\mathcal{M}D_{ij}] + \frac{m}{nk_BTd}\omega[V^2f^{(0)},\mathcal{M}D_{ij}]\right]$$ plus lon.

Calcul de V_k^{*a} :

$$\int_{\mathbb{R}^d}d\underline{v}\,S_i(\underline{v})\,L_a[S_i] \overset{3.3}{=} \sigma^{d-1}\phi V_T\int_{\mathbb{R}^{2d}}d\underline{v}_1d\underline{v}_2\,f^{(0)}(\underline{V}_1)\mathcal{M}(\underline{V}_2)\,S_i(V_2)\left[S_i(V_1)+S_i(V_2)\right]$$

$$=\sigma^{d-1}\phi V_T\int_{\mathbb{R}^{2d}}d\underline{v}_1d\underline{v}_2\frac{n}{V_T^d}\frac{1}{\pi^{d/2}}e^{-V_1^2/V_T^2}\frac{n}{V_T^d}\frac{1}{\pi^{d/2}}e^{-V_2^2/V_T^2}S_i(V_2)\left[S_i(V_1)+S_i(V_2)\right]$$

$$=\sigma^{d-1}\phi V_T\frac{n^2}{\pi^d}\frac{m^2}{4}V_T^6\int_{\mathbb{R}^{2d}}dc_1dc_2\,e^{-c_1^2}e^{-c_2^2}\left(c_2^2-\frac{d+2}{2}\right)\left[\left(c_1^2-\frac{d+2}{2}\right)\cancel{c_{1i}c_{2i}}^{\,0}+c_2^2\left(c_2^2-\frac{d+2}{2}\right)\right]$$

$$=\sigma^{d-1}\phi V_T^7\frac{n^2}{\pi^d}\frac{m^2}{4}\int_{\mathbb{R}^{2d}}dc_1dc_2\,e^{-c_1^2}e^{-c_2^2}c_2^2\left[c_2^4-\frac{d+2}{2}c_2^2\cdot 2+\frac{(d+2)^2}{4}\right]$$

$$=\sigma^{d-1}\phi V_T^7\frac{n^2}{\pi^d}\frac{m^2}{4}\pi^{d/2}\left[\int_{\mathbb{R}^d}dc\,e^{-c^2}c_2^6-(d+2)\int_{\mathbb{R}^d}dc\,e^{-c^2}c_2^4+\frac{(d+2)^2}{4}\int_{\mathbb{R}^d}dc\,e^{-c^2}c_2^2\right]$$

$$=\sigma^{d-1}\phi V_T^7\frac{n^2}{\pi^{d/2}}\frac{m^2}{4}\pi^{d/2}\left[\frac{\Gamma(\frac{d+6}{2})}{\Gamma(d/2)}-(d+2)\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}+\frac{(d+2)^2}{4}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}\right]$$

$$=\sigma^{d-1}\phi V_T^7\frac{n^2}{\cancel{\pi^{d/2}}}\frac{m^2}{4}\cancel{\pi^{d/2}}\left[\frac{d+4}{2}\frac{d+2}{2}\frac{d}{2}-(d+2)\frac{d+2}{2}\frac{d}{2}+\frac{(d+2)^2}{4}\frac{d}{2}\right]$$

$$=\sigma^{d-1}\phi V_T^7\frac{n^2m^2}{4}\frac{d+2}{2}\frac{d}{2}\underbrace{\left[\frac{d+4}{2}-d-2+\frac{d+2}{2}\right]}_{=\frac{1}{2}(\cancel{d}+4-\cancel{2d}-4+\cancel{d}+2)\ =\ \frac{1}{2}(2)=1}$$

$$=\sigma^{d-1}\phi V_T^7\frac{n^2m^2}{16}(d+2)d$$

$$\frac{2m\beta^3}{d(d+2)nV_0}\int_{\mathbb{R}^d}d\underline{v}\,S_i(\underline{v})\,L_a[S_i] = \frac{2m\beta^3}{\cancel{d(d+2)}\cancel{n}V_0}\sigma^{d-1}\phi V_T^7\frac{n^{\cancel{2}}m^2}{16}\cancel{d(d+2)}$$

$$= \frac{\cancel{\chi}\, \cancel{m^3} \cancel{\beta^3}}{\cancel{\chi} V_0} \sigma^{d-1} \phi\, V_T \left(\frac{\cancel{\chi}}{\cancel{\beta m}}\right)^3 n$$

$$= \frac{\sigma^{d-1} \phi\, V_T n}{V_0}$$

$$= \cancel{\sigma^{d-1}} \phi \cancel{V_T \chi} \frac{1}{\cancel{\chi \sigma^{d-1}}} \frac{d+2}{8} \frac{\sqrt{2}\,\Gamma(d/2)}{\cancel{V_T}\, \pi^{(d-1)/2}}$$

$$= \phi \frac{d+2}{8} \frac{\Gamma(d/2)\sqrt{2}}{\pi^{(d-1)/2}}$$

<u>Calcul de V_K^{*c}:</u>

$$\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v}) L_c[\mathcal{M} S_i] \overset{3.4}{=} -\sigma^{d-1} \frac{\phi V_T}{S_d} \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(v_1) \mathcal{M}(\underline{v}_2) S_i(\underline{v}_1) \int d\hat{\sigma}\, (b-1)[S_i(v_1) + S_i(v_2)]$$

$; S_i(V) = \left(\frac{m}{2} V^2 - \frac{d+2}{2} k_B T\right) V_i \quad ; V_T^2 = \frac{2}{\beta m}$

$$= -\sigma^{d-1} \frac{\phi V_T}{S_d} \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1) \mathcal{M}(\underline{v}_2) S_i(v_2) \int d\hat{\sigma}\, (b-1) \frac{m}{2}\left[V_1^2 V_{1i} + V_2^2 V_{2i} - \frac{d+2}{2} V_T^2 (V_{1i} + V_{2i})\right]$$

$= \frac{m}{2}\left(V^2 - \frac{d+2}{2} \frac{2}{m\beta}\right) V_i$

$= \frac{m}{2}\left(V^2 - \frac{d+2}{2} V_T^2\right) V_i$

$$= -\sigma^{d-1} \frac{\phi V_T}{S_d} \frac{m}{2} \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1) \mathcal{M}(\underline{v}_2) S_i(v_2) \left[\int d\hat{\sigma}\, (b-1)[V_1^2 V_{1i} + V_2^2 V_{2i}] - \frac{d+2}{2} V_T^2 \int d\hat{\sigma}\, (b-1)(V_{1i} + V_{2i}) \right]$$

avec:

$$(b-1)(V_{1i} + V_{2i}) = \cancel{V_{1i}} - (\cancel{V_{1}}\cdot\hat{\sigma})\sigma_i + \cancel{V_{2i}} + (\cancel{V_{2}}\cdot\hat{\sigma})\sigma_i - \cancel{V_{1i}} - \cancel{V_{2i}} = 0$$

$$(b-1)(V_1^2 V_{1i} + V_2^2 V_{2i}) = (g\cdot\hat{\sigma})^2 (V_{1j} + V_{2j})(\delta_{ij} + 2\sigma_i \sigma_j) - (g\cdot\hat{\sigma})(V_1^2 \delta_{ij} + 2 V_{1i} V_{1j} - V_2^2 \delta_{ij} - 2 V_{2i} V_{2j})\sigma_j$$

$$\int d\hat{\sigma}\, (b-1)(V_1^2 V_{1i} + V_2^2 V_{2i}) = (V_{1j} + V_{2j})\left[\delta_{ij}\, 2\beta_2 g^2 + 2\frac{2\beta_2}{d+2}(2 g_i g_j + g^2 \delta_{ij})\right] - (V_1^2 \delta_{ij} + 2 V_{1i} V_{1j} - V_2^2 \delta_{ij} - 2 V_{2i} V_{2j}) 2\beta_2 g_j$$

Ainsi:

$$\cancel{\int_{\mathbb{R}^d} d\underline{v}\, S_i(\underline{v}) L_c[\mathcal{M} S_i] = -\sigma^{d-1} \frac{\phi V_T}{S_d} \frac{m}{2} \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(v_1) \mathcal{M}(v_2) \frac{m}{2}\left(V_2^2 - \frac{d+2}{2} V_T^2\right) V_{2i} \left[2\beta_2 V_{1i} g^2 + 2\beta_2 V_{2i} g^2 + \frac{8\beta_2}{d+2} g_i (\underline{V}_1 \cdot g) + \frac{8\beta_2}{d+2} g_i (V_2 \cdot g) \ldots\right]}$$

$$\cancel{= -\sigma^{d-1} \frac{\phi V_T}{S_d} \frac{m^2}{4} \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(v_1) \mathcal{M}(v_2)\left(V_2^2 - \frac{d+2}{2} V_T^2\right)\left[-4\beta_2 \frac{1}{d} V_1^2 V_2^2 + 2\beta_2 V_2^2 (V_1^2 + V_2^2) + \frac{8\beta_2}{d+2}\left(-\frac{d+1}{d}\right) V_1^2 V_2^2\right]}$$

$$V_{2i} \int d\hat{\sigma}\, (b-1)(V_1^2 V_{1i} + V_2^2 V_{2i}) = V_{2i}\left[2\beta_2 g^2 (V_{1i} + V_{2i}) + \frac{4\beta_2}{d+2}\left(2 g_i (V_1\cdot g) + 2 g_i (V_2 \cdot g) + g^2 V_{1i} + g^2 V_{2i}\right) - 2\beta_2 \left(V_1^2 g_i + 2 V_{1i}(V_1\cdot g) - V_2^2 g_i - 2 V_{2i}(V_2\cdot g)\right)\right]$$

$$= 2\beta_2 g^2 [(V_1\cdot V_2) + V_2^2] + \frac{4\beta_2}{d+2}\left[2(g\cdot V_2)(V_1\cdot g) + 2(g\cdot V_2)(V_2\cdot g) + g^2 (V_1\cdot V_2) + g^2 V_2^2\right]$$
$$- 2\beta_2 \left[(V_1^2 - V_2^2)(g\cdot V_2) + 2(V_1\cdot V_2)(V_1\cdot g) - 2(V_2)^2 (V_2\cdot g)\right]$$

$; g^2 = V_1^2 + V_2^2 - 2(V_1\cdot V_2)$

$$= 2\beta_2\left[-2(V_1\cdot V_2)^2 + V_1^2 V_2^2 + V_2^4\right] + \frac{4\beta_2}{d+2}\left[2((V_1 - V_2)\cdot V_2)((V_1 - V_2)\cdot V_1) + 2((V_1 - V_2)\cdot V_2)^2 - 2(V_1\cdot V_2)^2 + V_1^2 V_2^2 + V_2^4\right]$$
$$- 2\beta_2\left[(V_1^2 - V_2^2)((V_1 - V_2)V_2) + 2(V_1\cdot V_2)(V_1(V_1 - V_2)) - 2V_2^2 (V_2\cdot(V_1 - V_2))\right]$$

$$= 2\beta_2\left[-\frac{2}{d}v_1^2v_2^2 + v_1^2v_2^2 + v_2^4\right] + \frac{4\beta_2}{d+2}\left[2(v_1\cdot v_2 - v_2^2)(v_1^2 - v_1\cdot v_2) + 2(v_1\cdot v_2 - v_2^2)^2 - \frac{2}{d}v_1^2v_2^2 + v_1^2v_2^2 + v_2^4\right]$$

$$-2\beta_2\left[(v_1^2-v_2^2)(\cancel{v_1\cdot v_2}^{\,0} - v_2^2) + 2(v_1\cdot v_2)(\cancel{v_1^2}^{\,0} - v_1\cdot v_2) - 2v_2^2(\cancel{v_2\cdot v_1}^{\,0} - v_2^2)\right]$$

$$= 2\beta_2\left[v_1^2v_2^2\left(1-\frac{2}{d}\right) + v_2^4 - (v_1^2-v_2^2)(-v_2^2) + 2(v_1\cdot v_2)^2 - 2v_2^4\right]$$

$$+\frac{4\beta_2}{d+2}\left[2v_1^2(\cancel{v_1\cdot v_2})^{0} - 2(v_1\cdot v_2)^2 - 2v_1^2v_2^2 + 2v_2^2(\cancel{v_1\cdot v_2})^{0} + 2\left((v_1\cdot v_2)^2 + v_2^4 - 2v_2^2\cancel{(v_1\cdot v_2)}^{0}\right) - \frac{2}{d}v_1^2v_2^2 + v_2^4 + v_1^2v_2^2\right]$$

$$= 2\beta_2\left[v_1^2v_2^2\frac{d-\cancel{2}}{d} + \cancel{v_2^4} + v_1^2v_2^2 - \cancel{v_2^4} + \cancel{\frac{2}{d}v_1^2}v_2^2 - 2v_2^4\right] + \frac{4\beta_2}{d+2}\left[-\cancel{\frac{2}{d}v_1^2}v_2^2 - \cancel{2}v_1^2v_2^2 + \cancel{\frac{2}{d}v_1^2}v_2^2 + 2v_2^4 - \frac{2}{d}v_1^2v_2^2 + v_2^4 + \cancel{v_1^2}v_2^2\right]$$

$$= 2\beta_2\left[2v_1^2v_2^2 - 2v_2^4\right] + \frac{4\beta_2}{d+2}\left[-v_1^2v_2^2 + 2v_2^4 - \frac{2}{d}v_1^2v_2^2 + v_2^4\right]$$

$$= 4\beta_2\left[v_1^2v_2^2 - v_2^4\right] + \frac{4\beta_2}{d+2}\left[+v_1^2v_2^2\frac{-2+d}{d} + 3v_2^4\right]$$

$$= \frac{4\beta_2}{d+2}\left[v_1^2v_2^2(d+2) - (d+2)v_2^4 + v_1^2v_2^2\frac{d-2}{d} + 3v_2^4\right]$$

$$= \frac{4\beta_2}{d+2}\left[v_1^2v_2^2\frac{d(d+2)+(d-2)}{d} + v_2^4(3-2-d)\right]$$

$$= \frac{4\beta_2}{d+2}\left[v_1^2v_2^2\frac{d(d+2)+(d-2)}{d} - (d-1)v_2^4\right]$$

Ainsi:

$$\int_{\mathbb{R}^d} dv\, S_i(V)\, L_c[MS_i] = -\sigma^{d-1}\frac{\phi v_T}{S_d}\frac{m^2}{4}v_T^6\int_{\mathbb{R}^{2d}} dc_1dc_2\,\frac{n}{\pi^{d/2}}e^{-c_1^2}\frac{n}{\pi^{d/2}}e^{-c_2^2}\left(c_2^2-\frac{d+2}{2}\right)\frac{4\beta_2}{d+2}\left[c_1^2c_2^2\frac{d(d+2)+(d-2)}{d} - (d-1)c_2^4\right]$$

$$= -\sigma^{d-1}\frac{\phi v_T^7}{S_d}\frac{m^2}{4}\frac{n^2}{\pi^d}\frac{4\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2}\left[c_1^2c_2^4\frac{d(d+2)+(d-1)}{d} - (d-1)c_2^6 - c_1^2c_2^2\frac{d+2}{2}\frac{d(d+2)+(d-1)}{d} + \frac{d+2}{2}(d-1)c_2^4\right]$$

~~On peut aussi reprendre les calculs de ...~~ ...etc...

ETC. repris plus loin.

$V_K^{*a'}$:

$$V_K^{*a'} = \frac{2m\beta^3}{d(d+2)nv_0}K_{ij}\int_{\mathbb{R}^d} dv\, f^{(0)}(v)\frac{\partial S_i}{\partial V_j}$$

$$= \frac{2m\beta^3}{d(d+2)nv_0}K_{ij}\int_{\mathbb{R}^d} dv\,\frac{n}{v_T^d}\frac{1}{\pi^{d/2}}e^{-V^2/v_T^2}\frac{m}{2}\left[2V_iV_j + \delta_{ij}\left(V^2-\frac{d+2}{2}v_T^2\right)\right]$$

$$= \frac{2m\beta^3}{d(d+2)nv_0}K_{ij}\frac{n}{v_T^d}v_T^d\frac{1}{\pi^{d/2}}\frac{m}{2}v_T^2\int_{\mathbb{R}^d} dc\, e^{-c^2}\left[2c_ic_j + \delta_{ij}\left(c^2-\frac{d+2}{2}\right)\right]$$

$$= \frac{\cancel{2}m\beta^3}{d(d+2)\cancel{nv_0}}\cancel{\frac{n}{v_T^d}}\frac{\cancel{v_T^d}}{\pi^{d/2}}\frac{m}{\cancel{2}}v_T^2K_{ij}\left[\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2}2c_ic_j}_{=2\delta_{ij}\pi^{d/2}\frac{d}{2d}\frac{\Gamma(d/2)}{\Gamma(d/2)} = \delta_{ij}\pi^{d/2}} + \delta_{ij}\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2}c^2}_{=\pi^{d/2}\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma(d/2)} = \pi^{d/2}\frac{d}{2}} - \frac{d+2}{2}\delta_{ij}\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2}}_{=\pi^{d/2}}\right]$$

$$= \frac{m\beta^3}{d(d+2)}mv_T^2K_{ij}\left[\delta_{ij} + \frac{d}{2}\delta_{ij} - \frac{d+2}{2}\delta_{ij}\right]$$

$$= \frac{m\beta^3}{d(d+2)}mv_T^2\,\mathrm{Tr}(K)\underbrace{\left[1+\frac{d}{2}-\frac{d+2}{2}\right]}_{=\frac{d+2}{2}-\frac{d+2}{2}} = 0\ !$$

ici aussi!

$$\Omega[MD_{ij}] = \frac{2}{n}\omega[f^{(0)}, MD_{ij}]\, f^{(0)} + T\frac{\partial f^{(0)}}{\partial T}[\dots]$$

$$\frac{\partial f^{(0)}}{\partial T} = \frac{\partial}{\partial T}\left(\frac{n}{v_T^d} e^{-v^2/v_T^2}\right) \quad ; v_T = \sqrt{\frac{2kT}{m}} \; ; \partial_T v_T = \sqrt{\frac{2k}{m}}\frac{T^{1/2}}{T}\frac{1}{2} = \frac{v_T}{2T}$$

$$= -d\frac{n}{v_T^{d+1}}(\partial_T v_T)\, e^{-v^2/v_T^2} + \frac{n}{v_T^d}\left(-\frac{v^2}{v_T^2}\right)e^{-v^2/v_T^2}(-2)\frac{1}{v_T}\partial_T v_T$$

$$= -\frac{d}{v_T} f^{(0)}\frac{v_T}{2T} + 2\frac{v^2}{v_T^2} f^{(0)}\frac{1}{v_T}\frac{v_T}{2T}$$

$$= -\frac{d}{2T} f^{(0)} + \frac{v^2}{v_T^2}\frac{1}{Tv_T^2} f^{(0)}$$

$$= f^{(0)}\left[\frac{v^2}{v_T^2}\frac{1}{T} - \frac{d}{2T}\right]$$

$$= f^{(0)}\frac{1}{T}\left[\frac{v^2}{v_T^2} - \frac{d}{2}\right]$$

$\Rightarrow$

$$\Omega[MD_{ij}] = \frac{2}{n}\omega[f^{(0)}, MD_{ij}]\, f^{(0)} + f^{(0)}\left[\frac{v^2}{v_T^2} - \frac{d}{2}\right][\dots]$$

$$= f^{(0)}K_{ij} + f^{(0)}\left[\frac{v^2}{v_T^2} - \frac{d}{2}\right]L_{ij}$$

$$\frac{\beta^2}{(d+2)(d-1)nv_0}\int_{\mathbb{R}^d} dv\, D_{ij}(v)\,\Omega[MD_{ij}] = \frac{\beta^2}{(d+2)(d-1)nv_0}\left[K_{ij}\int_{\mathbb{R}^d} dv\, D_{ij}(v) f^{(0)} + L_{ij}\int_{\mathbb{R}^d} dv\, f^{(0)}\left(\frac{v^2}{v_T^2} - \frac{d}{2}\right)D_{ij}\right]$$

$$= \frac{\beta^2 m}{(d+2)(d-1)nv_0}\left[K_{ij}\int_{\mathbb{R}^d} dv\, f^{(0)}(v)\left(v_iv_j - \frac{1}{d}v^2\delta_{ij}\right) + L_{ij}\int_{\mathbb{R}^d} dv\, f^{(0)}(v)\left(v_iv_j - \frac{1}{d}v^2\delta_{ij}\right)\left(\frac{v^2}{v_T^2} - \frac{d}{2}\right)\right]$$

$$= \frac{\beta^2 m}{(d+2)(d-1)nv_0}\left[K_{ij}\frac{n}{\cancel{v_T^d}}\frac{\cancel{v_T^d}}{\pi^{d/2}}\int dc\, e^{-c^2} v_T^2\left(c_ic_j - \frac{1}{d}c^2\delta_{ij}\right) + L_{ij}\frac{n}{v_T^d}\frac{v_T^d}{\pi^{d/2}}v_T^2\int dc\, e^{-c^2}\left(c_ic_j - \frac{1}{d}c^2\delta_{ij}\right)\left(c^2 - \frac{d}{2}\right)\right]$$

$$= \frac{\beta^2 m}{(d+2)(d-1)nv_0}\frac{nv_T^2}{\pi^{d/2}}\left[K_{ij}\underbrace{\int dc\, e^{-c^2}c_ic_j}_{=\delta_{ij}\pi^{d/2}\frac{d}{2d}} - K_{ij}\frac{1}{d}\delta_{ij}\underbrace{\int dc\, e^{-c^2}c^2}_{=\pi^{d/2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}} + L_{ij}\int dc\, e^{-c^2}\left(c^2c_ic_j - \frac{d}{2}c_ic_j - \frac{1}{d}c^4\delta_{ij} + \frac{1}{2}c^2\delta_{ij}\right)\right]$$

$$= \frac{\beta^2 m}{(d+2)(d-1)nv_0}\frac{nv_T^2}{\pi^{d/2}}\left[\cancel{K_{ij}\delta_{ij}\frac{\pi^{d/2}}{2}} - \cancel{K_{ij}\delta_{ij}\frac{1}{d}\pi^{d/2}\frac{d}{2}} + L_{ij}\int dc\, e^{-c^2}c^2c_ic_j - \frac{d}{2}L_{ij}\underbrace{\int dc\, e^{-c^2}c_ic_j}_{=\delta_{ij}\pi^{d/2}\frac{d}{2d}} - \frac{1}{d}\delta_{ij}L_{ij}\underbrace{\int dc\, e^{-c^2}c^4}_{=\pi^{d/2}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}} + \frac{1}{2}\delta_{ij}L_{ij}\underbrace{\int dc\, e^{-c^2}c^2}_{=\pi^{d/2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}}\right]$$

$$= \frac{\beta^2 m}{(d+2)(d-1)nv_0}\frac{nv_T^2}{\pi^{d/2}}L_{ij}\left[\underbrace{\int dc\, e^{-c^2}c^2c_ic_j}_{=\pi^{d/2}\frac{d+2}{2d}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}\delta_{ij}} - \frac{d}{2}\delta_{ij}\pi^{d/2}\frac{1}{2} - \frac{1}{d}\delta_{ij}\pi^{d/2}\frac{d+2}{2}\frac{d}{2} + \frac{1}{2}\delta_{ij}\pi^{d/2}\frac{d}{2}\right]$$

$$= \frac{\beta^2 m}{(d+2)(d-1)nv_0}\frac{nv_T^2}{\pi^{d/2}}L_{ij}\delta_{ij}\pi^{d/2}\left[\cancel{\frac{d+2}{2d}\frac{d}{2}} - \cancel{\frac{d}{4}} - \cancel{\frac{d+2}{2}} + \cancel{\frac{d}{4}}\right]$$

$$= 0 \quad (!!!)$$

Pour les collisions, on peut réutiliser un résultat de Santos (on pourrait aussi le calculer) :

$$① \quad \frac{\omega}{2n\Omega_d}\int dv_1dv_2\, f(v_1)f(v_2)\int d\hat\sigma\,(b-1)[V_1V_1+V_2V_2] = -\omega\frac{1+\alpha}{4nd}\int dv_1\int dv_2\, f(v_1)f(v_2)\left[\frac{1-\alpha}{d}g^2\mathbb{1}+2\frac{d+1-\alpha}{d+2}\left(gg-\frac{1}{d}g^2\mathbb{1}\right)\right]$$

$$② \quad \frac{\omega}{2n\Omega_d}\int dv_1dv_2\, f(v_1)f(v_2)\int d\hat\sigma\,(b-1)(V_1^2V_1+V_2^2V_2) = \frac{\omega(1+\alpha)}{4nd}\int dv_1dv_2\, f(v_1)f(v_2)\left[\frac{1+\alpha}{2(d+2)}(V_1+V_2)[(d+4)g^2\mathbb{1}+4gg] - g\left(V_1^2\mathbb{1}-V_2^2\mathbb{1}+2V_1V_1-2V_2V_2\right)\right]$$

Dans notre cas il suffira pour ① de remplacer $f(v_2)$ par $M(v_2)D_{ij}(v_2)$ et pour ② $f(v_2)$ par $M(v_2)S_i(v_2)$. Particulièrement :

①:

$$\int dv_1\int dv_2\, f(v_1)f(v_2)\int d\hat\sigma\,(b-1)[V_1V_1+V_2V_2] = -\frac{\cancel{2n}\Omega_d}{\cancel{\omega}}\cancel{\omega}\frac{\cancel{2}}{\cancel{4n}d}\int dv_1dv_2\, f(v_1)f(v_2)\left[0+2\frac{d}{d+2}\left(gg-\frac{1}{d}g^2\mathbb{1}\right)\right]$$

$$\Rightarrow \int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\int d\hat\sigma\,(b-1)[V_{1i}V_{1j}+V_{2i}V_{2j}] = -\frac{\Omega_d}{\cancel{d}}2\frac{\cancel{d}}{d+2}\int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\left(g_ig_j-\frac{1}{d}g^2\delta_{ij}\right)$$

$$\Rightarrow \int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\int d\hat\sigma\,\underbrace{(b-1)\; m\left[V_{1i}V_{1j}+V_{2i}V_{2j}\right]}_{=(b-1)[D_{ij}(v_1)+D_{ij}(v_2)]} = -\frac{2\Omega_d}{d+2}m\int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\left(g_ig_j-\frac{1}{d}g^2\delta_{ij}\right)$$

$$\Rightarrow \boxed{\int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\int d\hat\sigma(b-1)[D_{ij}(v_1)+D_{ij}(v_2)] = -\frac{2m\Omega_d}{d+2}\int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\left(g_ig_j-\frac{1}{d}g^2\delta_{ij}\right)}$$

$$② \quad \int dv_1dv_2\, f(v_1)f(v_2)\int d\hat\sigma\,\underbrace{(b-1)(V_1^2V_1+V_2^2V_2)} = \frac{\cancel{2n}\Omega_d}{\cancel{\omega}}\frac{\cancel{\omega}\,\cancel{(1+\alpha)}}{\cancel{4n}d}\int dv_1dv_2\, f(v_1)f(v_2)\left[\frac{2}{2(d+2)}(V_1+V_2)\left[(d+4)g^2\mathbb{1}+4gg\right] - g\left(V_1^2\mathbb{1}-V_2^2\mathbb{1}+2V_1V_1-2V_2V_2\right)\right]$$

$$= (b-1)\left(V_1^2V_1+V_2^2V_2-aV_1-bV_2\right)$$

$$= \frac{2}{m}(b-1)\left[\frac{m}{2}\left[\left(V_1^2-\frac{d+2}{2}v_T^2\right)V_1+\left(V_2^2-\frac{d+2}{2}v_T^2\right)\right]\right]$$

$$= \frac{2}{m}(b-1)\left[S(v_1)+S(v_2)\right]$$

$$\Rightarrow \int dv_1dv_2\, f(v_1)f(v_2)\int d\hat\sigma\,(b-1)[S(v_1)+S(v_2)] = \frac{m}{2}\frac{\Omega_d}{d}\int dv_1dv_2\, f(v_1)f(v_2)\left[\frac{V_1+V_2}{d+2}\left[(d+4)g^2\mathbb{1}+4gg\right] - g\left(V_1^2\mathbb{1}-V_2^2\mathbb{1}+2(V_1V_1)-2(V_2V_2)\right)\right]$$

√OK

$$\Rightarrow \boxed{\int dv_1dv_2\, f^{(0)}(v_1)M(v_2)S_i(v_2)\int d\hat\sigma\,(b-1)\left[S_i(v_1)+S_i(v_2)\right] = \frac{m}{2}\frac{\Omega_d}{d}\int dv_1dv_2\, f(v_1)M(v_2)S_i(v_2)\left[\frac{V_{1j}+V_{2j}}{d+2}\left[(d+4)g^2\delta_{ij}+4g_ig_j\right] - g_j\left((V_1^2-V_2^2)\delta_{ij}+2V_{1i}V_{1j}-2V_{2i}V_{2j}\right)\right]}$$

Avec ces résultats on peut calculer à moindre risque V_ℓ^{*c} et V_k^{*c} :

$$V_\ell^{*c}: \int_{\mathbb{R}^d} dv\, D_{ij}(v)L_c[MD_{ij}] = -\sigma^{d-1}\frac{\phi v_T}{\Omega_d}\int_{\mathbb{R}^{2d}} dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\int d\hat\sigma\,(b-1)\left[D_{ij}(v_1)+D_{ij}(v_2)\right]$$

$$\overset{①}{=} -\sigma^{d-1}\frac{\phi v_T}{\cancel{\Omega_d}}\left(-\frac{2m\cancel{\Omega_d}}{d+2}\right)\int dv_1dv_2\, f^{(0)}(v_1)M(v_2)D_{ij}(v_2)\left(g_ig_j-\frac{1}{d}g^2\delta_{ij}\right) \qquad (A)$$

$$V_k^{*c}: \int_{\mathbb{R}^d} dv\, S_i(v)L_c[MS_i] = -\sigma^{d-1}\frac{\phi v_T}{\Omega_d}\int_{\mathbb{R}^{2d}} dv_1dv_2\, f^{(0)}(v_1)M(v_2)S_i(v_2)\int d\hat\sigma\,(b-1)\left[S_i(v_1)+S_i(v_2)\right]$$

$$\overset{②}{=} -\sigma^{d-1}\frac{\phi v_T}{\Omega_d}\frac{m}{2}\frac{\Omega_d}{d}\int dv_1dv_2\, f(v_1)M(v_2)S_i(v_2)\left[\frac{V_{1j}+V_{2j}}{d+2}\left\{(d+4)g^2\delta_{ij}+4g_ig_j\right\} - g_j\left\{(V_1^2-V_2^2)\delta_{ij}+2V_{1i}V_{1j}-2V_{2i}V_{2j}\right\}\right] \qquad (B)$$

Suite du calcul :

$$V_\ell^{*c}: \quad D_{ij}(v_2)\left(g_ig_j-\frac{1}{d}g^2\delta_{ij}\right) = m\left(V_{2i}V_{2j}-\frac{1}{d}V_2^2\delta_{ij}\right)\left(g_ig_j-\frac{1}{d}g^2\delta_{ij}\right)$$

$$= m\left(V_{2i}V_{2j}-\frac{1}{d}V_2^2\delta_{ij}\right)\left[\underbrace{(V_{1i}-V_{2i})(V_{1j}-V_{2j})}_{=V_{1i}V_{1j}-V_{1i}V_{2j}-V_{2i}V_{1j}+V_{2i}V_{2j}} - \frac{1}{d}\delta_{ij}\left(V_1^2+V_2^2-2(V_1\cdot V_2)\right)\right]$$

$$= m\left(V_{2i}V_{2j}-\frac{1}{d}V_2^2\delta_{ij}\right)\left(V_{1i}V_{1j}-V_{1i}V_{2j}-V_{2i}V_{1j}+V_{2i}V_{2j}-\frac{1}{d}\delta_{ij}(V_1^2+V_2^2)-\frac{2}{d}\delta_{ij}(V_1\cdot V_2)\right)$$

$$= m \Big(V_{2i}V_{2j}V_{1i}V_{1j} - V_{2i}V_{2j}V_{1i}V_{2j} - V_{2i}V_{2j}V_{2i}V_{1j} + V_{2i}V_{2j}V_{2i}V_{2j}$$
$$- \frac{1}{d} V_{2i}V_{2j}\delta_{ij}(V_1^2+V_2^2) - \frac{2}{d} V_{2i}V_{2j}\delta_{ij}(V_1\cdot V_2) - \frac{1}{d}V_2^2\delta_{ij}V_{1i}V_{1j} - \frac{1}{d}V_2^2\delta_{ij}V_{1i}V_{2j}$$
$$+ \frac{1}{d}V_2^2\delta_{ij}V_{2i}V_{1j} - \frac{1}{d}V_2^2\delta_{ij}V_{2i}V_{2j} + \frac{1}{d^2}V_2^2\delta_{ij}\delta_{ij}(V_1^2+V_2^2) + \frac{2}{d^2}V_2^2\delta_{ij}\delta_{ij}(V_1\cdot V_2)\Big)$$

$$= m\Big((V_1\cdot V_2)^2 - \cancel{(V_1\cdot V_2)}^{0} V_2^2 - V_2^2\cancel{(V_1\cdot V_2)}^{0} + V_2^4 - \frac{1}{d}V_2^2(V_1^2+V_2^2) - \frac{2}{d}V_2^2\cancel{(V_1\cdot V_2)}^{0} - \frac{1}{d}V_2^2V_1^2$$
$$- \frac{1}{d}\cancel{V_2^2(V_1\cdot V_2)} + \frac{1}{d}\cancel{V_2^2(V_1\cdot V_2)} - \frac{1}{d}V_2^4 + \frac{1}{d^2}V_2^2(V_1^2+V_2^2)d + \frac{2}{d}V_2^2\cancel{(V_1\cdot V_2)}^{0}\Big)$$

$$= m\Big(\cancel{\frac{1}{d}V_1^2V_2^2} + V_2^4 - \cancel{\frac{1}{d}V_1^2V_2^2} - \frac{1}{d}V_2^4 - \cancel{\frac{1}{d}V_1^2V_2^2} - \cancel{\frac{1}{d}V_2^4} + \cancel{\frac{1}{d}V_1^2V_2^2} + \cancel{\frac{1}{d}V_2^4}\Big)$$

$$= m\,\frac{d-1}{d}V_2^4$$

$\Longrightarrow$

$$\int_{\mathbb{R}^d} dv\, D_{ij}(v)L_c[MD_{ij}] = +\sigma^{d-1}\varnothing v_T\,\frac{2m}{d+2}\,m\,\frac{d-1}{d}\int dv_1dv_2\, f^{(0)}(v_1)\,M(v_2)\,V_2^4$$

$$= \sigma^{d-1}\varnothing v_T\, 2m^2\frac{d-1}{d(d+2)}\left(\frac{n}{v_T^d}\right)^2\frac{1}{\pi^d}\,v_T^4\int dc_1dc_2\, e^{-c_1^2}e^{-c_2^2}c_2^4\, v_T^d$$

$$= \sigma^{d-1}\varnothing v_T^5\, 2m^2\frac{d-1}{d(d+2)}n^2\frac{1}{\pi^{d/2}}\underbrace{\int dc\, e^{-c^2}c^4}_{=\pi^{d/2}\frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)} = \pi^{d/2}\frac{d+2}{2}\frac{d}{2}}$$

$$= \sigma^{d-1}\varnothing v_T^5\, 2m^2\frac{d-1}{\cancel{d(d+2)}}n^2\cancel{\frac{d+2}{2}\frac{d}{2}}$$

$$\Longrightarrow \frac{\beta^2}{(d+2)(d-1)n\nu_0}\int_{\mathbb{R}^d} dv\, D_{ij}(v)L_c[MD_{ij}] = \frac{\cancel{\beta^2}}{(d+2)(d-1)\cancel{n}\nu_0}\sigma^{d-1}\varnothing v_T\left(\frac{\cancel{2}}{\cancel{\beta m}}\right)^2 2\cancel{m^2}\frac{d-1}{\cancel{4}}n^{\cancel{2}}$$

$$= \frac{2\sigma^{d-1}\varnothing v_T n}{(d+2)\cancel{(d-1)}\nu_0}\cancel{(d-1)}$$

$$= \frac{2}{d+2}\sigma^{d-1}\varnothing v_T n\frac{1}{\nu_0} \quad ; \quad \nu_0 = n\sigma^{d-1}\frac{8}{d+2}\frac{1}{v_2}\cancel{v_T}\,\pi^{(d-1)/2}\Big/\Gamma(d/2)$$

$$= \frac{2}{\cancel{d+2}}\cancel{\sigma^{d-1}}\varnothing\cancel{v_Tn}\,\frac{\cancel{(d+2)}\Gamma(d/2)\,v_2}{\cancel{n\sigma^{d-1}}\,8\,\cancel{v_2}\,\cancel{v_T}\,\pi^{(d-1)/2}}$$

$$\nu_0 = \frac{p^{(c)}}{\eta_0} = nk_BT\frac{8}{d+2}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}\frac{\sigma^{d-1}}{\sqrt{mk_BT}} = n\sigma^{d-1}\frac{8}{d+2}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}\frac{1}{v_2}\quad \underbrace{\sqrt{\frac{2k_BT}{m}}}_{=v_T}$$

$$= \frac{2\varnothing\,\Gamma(d/2)\,v_2}{8\,v_2\,\pi^{(d-1)/2}}$$

$$= \boxed{\frac{\varnothing\,\Gamma(d/2)\,v_2}{4\,\cancel{v_2}\,\pi^{(d-1)/2}}} = \nu_\eta^{*c}$$

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$$S_r(V) = \frac{m}{2}\left(V^2 - \frac{d+2}{2}v_T^2\right)$$

$$V_k^{*c}:\quad S_r(V_2)\Big[\frac{V_{1j}+V_{2j}}{d+2}\left\{(d+4)g^2\delta_{ij} + 4g_ig_j\right\} - g_j\left\{(V_1^2-V_2^2)\delta_{ij} + 2V_{1i}V_{1j} - 2V_{2i}V_{2j}\right\}\Big]$$

$$= S_r(V_2)\Big[\frac{d+4}{d+2}g^2(V_{1i}+V_{2i}) + \frac{4}{d+2}g_i\underbrace{g_jV_{1j}}_{=(g\cdot V_1)} + \frac{4}{d+2}g_i\underbrace{g_jV_{2j}}_{=(g\cdot V_2)} - (V_1^2-V_2^2)g_i - 2V_{1i}V_{1j}g_j + 2V_{2i}V_{2j}g_j\Big]$$

$$= S_r(V_2)\Big[\frac{d+4}{d+2}g^2(V_{1i}+V_{2i}) + \frac{4}{d+2}g_i(g\cdot V_1) + \frac{4}{d+2}g_i(g\cdot V_2) - (V_1^2-V_2^2)g_i - 2V_{1i}(V_1\cdot g) + 2V_{2i}(V_2\cdot g)\Big]$$

$$= \frac{m}{2}\left(V_2^2 - \frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}g^2(V_{1i}V_{2i}+V_{2i}V_{2i}) + \frac{4}{d+2}(g_iV_{2i})(g\cdot V_1) + \frac{4}{d+2}(V_{2i}g_i)(V_2\cdot g) - (V_1^2-V_2^2)g_iV_{2i}$$
$$- 2V_{1i}V_{2i}(V_1\cdot g) + 2V_{2i}V_{2i}(V_2\cdot g)\Big]$$

$$= \frac{m}{2}\left(V_2^2 - \frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}g^2(V_1\cdot V_2) + \frac{d+4}{d+2}g^2V_2^2 + \frac{4}{d+2}(g\cdot V_2)(g\cdot V_1) + \frac{4}{d+2}(g\cdot V_2)^2 - (V_1^2-V_2^2)(g\cdot V_2) - 2(V_1\cdot V_2)(V_1\cdot g)$$
$$+ 2V_2^2(V_2\cdot g)\Big]$$

$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}(v_1-v_2)^2(v_1\cdot v_2) + \frac{d+4}{d+2}(v_1^2+v_2^2-2v_1\cdot v_2)v_2^2 + \frac{4}{d+2}((v_1-v_2)v_2)((v_1-v_2)v_1)$$
$$+\frac{4}{d+2}((v_1-v_2)v_2)^2 - (v_1^2-v_2^2)((v_1-v_2)v_2) - 2(v_1\cdot v_2)(v_1(v_1-v_2))$$
$$+2v_2^2(v_2(v_1-v_2))\Big]$$

$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}(v_1^2+v_2^2-2v_1\cdot v_2)(v_1\cdot v_2) + \frac{d+4}{d+2}(v_1^2+v_2^2)v_2^2 + \frac{4}{d+2}(v_1\cdot v_2-v_2^2)(v_1^2-v_1\cdot v_2)$$
$$+\frac{4}{d+2}(v_1\cdot v_2-v_2^2)^2 - (v_1^2-v_2^2)(v_1\cdot v_2-v_2^2) - 2(v_1\cdot v_2)(v_1^2-v_1\cdot v_2)\Big]$$
$$+2v_2^2(v_2 v_1 - v_2^2)$$

$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}(-2)(v_1\cdot v_2)^2 + \frac{d+4}{d+2}(v_1^2v_2^2+v_2^4) + \frac{4}{d+2}\left(-(v_1\cdot v_2)^2 - v_1^2v_2^2\right)$$
$$+\frac{4}{d+2}\left((v_1\cdot v_2)^2+v_2^4\right) + v_2^2(v_1^2-v_2^2) + 2(v_1\cdot v_2)^2\Big]$$
$$-2v_2^4$$

$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[-2\frac{d+4}{d+2}\frac{1}{d}v_1^2v_2^2 + \frac{d+4}{d+2}v_1^2v_2^2 + \frac{d+4}{d+2}v_2^4 - \cancel{\frac{4}{d+2}\frac{1}{d}v_1^2v_2^2} - \frac{4}{d+2}v_1^2v_2^2$$
$$+\cancel{\frac{4}{d+2}\frac{1}{d}v_1^2v_2^2} + \frac{4}{d+2}v_2^4 + v_1^2v_2^2 - v_2^4 + \frac{2}{d}v_1^2v_2^2\Big]$$
$$-2v_2^4$$

$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\left\{-\frac{2}{d}\frac{d+4}{d+2} + \frac{d+4}{d+2} - \frac{4}{d+2} + 1 + \frac{2}{d}\right\}$$
$$+v_2^4\left\{\frac{d+4}{d+2} + \frac{4}{d+2} - 1 - 2\right\}\Big]$$

$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\,\frac{1}{d(d+2)}\left[-2(d+4)+d(d+4)-4d+d(d+2)+2(d+2)\right]$$
$$+v_2^4\,\frac{1}{d+2}\left[d+4+4-3(d+2)\right]\Big\}$$

$$= \frac{m}{2(d+2)}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\,\frac{1}{d}\left(-\cancel{2d}-8+d^2+\cancel{4d}-\cancel{4d}+d^2+\cancel{2d}+2d+4\right)$$
$$+v_2^4(\ldots)\Big]$$
$$(d+8-3d-6) = (-2d+2) = 2(1-d) = -2(d-1)$$

$$= \frac{m}{2(d+2)}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\,\frac{1}{d}\underbrace{\left(2d^2+2d-4\right)}_{\substack{=2(d^2+d-2)\\ =2(d+2)(d-1)}} -2v_2^4(d-1)\Big]$$

$$= \frac{m}{2(d+2)}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\,\frac{2}{d}(d+2)(d-1) - 2v_2^4(d-1)\Big]$$

VERIFICATION:

$$S_i(v_2)\Big[\frac{v_{1j}+v_{2j}}{d+2}\left\{(d+4)g^2\delta_{ij} + 4g_ig_j\right\} - g_j\left\{(v_1^2-v_2^2)\delta_{ij} + 2v_{1i}v_{1j} - 2v_{2i}v_{2j}\right\}\Big]$$
$$= S_i(v_1)\Big[\frac{d+4}{d+2}g^2(v_{1i}+v_{2i}) + \frac{4}{d+2}(v_{1j}g_jg_i + v_{2j}g_jg_i) - (v_1^2-v_2^2)g_i - 2v_{1i}(v_1\cdot g) + 2v_{2i}(v_2\cdot g)\Big]$$
$$= S_i(v_2)\Big[\frac{d+4}{d+2}g^2(v_{1i}+v_{2i}) + \frac{4}{d+2}\left((v_1\cdot g)g_i + (v_2\cdot g)g_i\right) - (v_1^2-v_2^2)g_i - 2v_{1i}(v_1\cdot g) + 2v_{2i}(v_2\cdot g)\Big]$$
$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}g^2\left((v_1\cdot v_2)+v_2^2\right) + \frac{4}{d+2}\left((v_1\cdot g)(v_2\cdot g) + (v_2\cdot g)^2\right) - (v_1^2-v_2^2)(v_2\cdot g) - 2(v_1\cdot g)(v_2\cdot v_1) + 2v_2^2(v_2\cdot g)\Big]$$
$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}(v_1^2+v_2^2-2v_1\cdot v_2)\left((v_1\cdot v_2)+v_2^2\right) + \frac{4}{d+2}\left((v_1\cdot(v_1-v_2))(v_2(v_1-v_2)) + (v_2(v_1-v_2))^2\right)$$
$$-(v_1^2-v_2^2)(v_2(v_1-v_2)) - 2(v_1(v_1-v_2))(v_1\cdot v_2) + 2v_2^2(v_2\cdot(v_1-v_2))\Big]$$
$$= \frac{m}{2}\left(v_2^2-\frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}\left(v_1^2v_2^2+v_2^4-2(v_1\cdot v_2)^2\right) + \frac{4}{d+2}\left((v_1^2-v_1\cdot v_2)(v_1\cdot v_2-v_2^2) + (v_1\cdot v_2-v_2^2)^2\right)$$
$$-(v_1^2-v_2^2)(v_1\cdot v_2-v_2^2) - 2(v_1^2-v_1\cdot v_2)(v_1\cdot v_2) + 2v_2^2(v_1\cdot v_2-v_2^2)\Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}\left(v_1^2 v_2^2 + v_2^4 - \frac{2}{d}v_1^2v_2^2\right) + \frac{4}{d+2}\left(-v_1^2v_2^2 - \cancel{\frac{1}{d}v_1^2v_2^2}\right) + \frac{4}{d+2}\left(\cancel{\frac{1}{d}v_1^2v_2^2} + v_2^4\right)$$
$$ - (v_1^2 - v_2^2)\left(-v_2^2\right) - 2\left(-\frac{1}{d}\right)v_1^2v_2^2 - 2v_2^4 \Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}\left[\frac{d-2}{d}v_1^2v_2^2 + v_2^4\right] + \frac{4}{d+2}\left(-v_1^2v_2^2 + v_2^4\right) + v_2^2(v_1^2 - v_2^2) + \frac{2}{d}v_1^2v_2^2 - 2v_2^4\Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\Big[\frac{d+4}{d+2}\frac{d-2}{d}v_1^2v_2^2 + \underbrace{\frac{d+4}{d+2}v_2^4} - \frac{4}{d+2}v_1^2v_2^2 + \underbrace{\frac{4}{d+2}v_2^4} + v_1^2v_2^2 \underbrace{- v_2^4} + \frac{2}{d}v_1^2v_2^2 \underbrace{- 2v_2^4}\Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\left(\frac{d+4}{d+2}\frac{d-2}{d} - \frac{4}{d+2} + 1 - \frac{2}{d}\right) + v_2^4\left(\frac{d+4}{d+2} + \frac{4}{d+2} - 1 - 2\right)\Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\frac{1}{d+2}\Big[v_1^2v_2^2\frac{1}{d}\left((d+4)(d-2) - 4d + d(d+2) + 2(d+2)\right) + v_2^4\left(d+4+4-(d+2)\,3\right)\Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\frac{1}{d+2}\Big[v_1^2v_2^2\frac{1}{d}\left(d^2 \cancel{-2d} \cancel{+4d} - 8 \cancel{-4d} + d^2 + \cancel{2d} + 2d + 4\right) + v_2^4\left(d - 3d + 4 + 4 - 6\right)\Big]$$

$$= \frac{m}{2}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\frac{1}{d+2}\Big[v_1^2v_2^2\frac{1}{d}\left(2d^2 + 2d - 4\right) + v_2^4\left(-2d + 2\right)\Big]$$

$$= \frac{m}{2(d+2)}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\frac{2}{d}\left(d^2 + d - 2\right) - 2v_2^4(d-1)\Big]$$

$$= \frac{m}{2(d+2)}\left(v_2^2 - \frac{d+2}{2}v_T^2\right)\Big[v_1^2v_2^2\frac{2}{d}(d+2)(d-1) - 2(d-1)v_2^4\Big]$$

Ok.

$$= \frac{m}{2(d+2)}\Big[v_1^2v_2^4\frac{2}{d}(d+2)(d-1) - 2(d-1)v_2^6 - v_1^2v_2^2v_T^2\frac{d+2}{\cancel{2}}\frac{\cancel{2}}{d}(d+2)(d-1) + \frac{d+2}{2}2(d-1)v_T^2v_2^4\Big]$$

$$= \frac{m}{2(d+2)}\Big[v_1^2v_2^4\frac{2(d+2)(d-1)}{d} - v_2^6\,2(d-1) - v_1^2v_2^2v_T^2\frac{(d+2)^2(d-1)}{d} + v_2^4v_T^2(d+2)(d-1)\Big]$$

$\Rightarrow$

$$\int dv_1 dv_2\, f^{(1)}(v_1)M(v_2)S_i(v_2)\int d\hat{\sigma}\,(b-1)\,\left[S_i(v_1) + S_i(v_2)\right]$$

$$= \frac{m}{2(d+2)}\int dv_1 dv_2\, f^{(1)}(v_1)M(v_2)\left[\ \dots\ \right]$$

$$= \frac{m}{2(d+2)}\frac{n^2}{\cancel{v_T^{2d}}}\frac{1}{\pi^d}\cancel{v_T^{2d}}v_T^6\int dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\Big[c_1^2c_2^4\frac{2(d+2)(d-1)}{d} - c_2^6\,2(d-1) - c_1^2c_2^2\frac{(d+2)^2(d-1)}{d} + c_2^4(d+2)(d-1)\Big]$$

$$= \frac{m}{2(d+2)}\frac{n^2}{\pi^d}v_T^6\Big[\int dc_1 e^{-c_1^2}c_1^2\int dc_2 e^{-c_2^2}c_2^4\frac{2(d+2)(d-1)}{d} - 2(d-1)\pi^{d/2}\int dc\, e^{-c^2}c^6 - \frac{(d+2)^2(d-1)}{d}\left(\int dc\, e^{-c^2}c^2\right)^2$$
$$ + (d+2)(d-1)\pi^{d/2}\int dc\, e^{-c^2}c^4\Big]$$

$$= \frac{m}{2(d+2)}\frac{n^2}{\pi^d}v_T^6\Big[\pi^{d/2}\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma(d/2)}\pi^{d/2}\frac{\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d/2)}\frac{2(d+2)(d-1)}{d} - 2(d-1)\pi^{d/2}\pi^{d/2}\frac{\Gamma\left(\frac{d+6}{2}\right)}{\Gamma(d/2)} - \frac{(d+2)^2(d-1)}{d}\pi^d\left[\frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma(d/2)}\right]^2$$
$$ + (d+2)(d-1)\pi^{d/2}\pi^{d/2}\frac{\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d/2)}\Big]$$

$$= \frac{m}{2\cancel{(d+2)}}n^2v_T^6\Big[\frac{\cancel{d}}{2}\frac{\cancel{d+2}}{2}\frac{2(d+2)(d-1)}{\cancel{d}} - \cancel{2}(d-1)\frac{d+4}{\cancel{2}}\frac{\cancel{d+2}}{2}\frac{d}{2} - \frac{(d+2)^{\cancel{2}}(d-1)}{d}\left(\frac{d}{2}\right)^2 + \cancel{(d+2)}(d-1)\frac{d+2}{2}\frac{d}{2}\Big]$$

$$= \frac{m}{8}n^2v_T^6\Big[2(d+2)(d-1) - (d+4)(d-1)d - (d+2)\cancel{(d-1)}d + \cancel{(d-1)}(d+2)d\Big]$$

$$= m\frac{d-1}{8}n^2v_T^6\Big[2(d+2) - d(d+4)\Big]$$

$$= \frac{d-1}{8}mn^2v_T^6\Big[\underbrace{2d + 4 - d^2 - 4d}_{= -d^2 - 2d + 4}\Big]$$

$$= -\frac{(d^2+2d-4)(d-1)}{8}mn^2v_T^6$$

VERIFICATION :

$$\frac{m}{2(d+2)}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}\int dv_1 dv_2\, e^{-v_1^2/V_T^2} e^{-v_2^2/V_T^2}\,[\dots]$$

$$= \frac{m}{2(d+2)}\frac{n^2}{V_T^{2d}}\frac{1}{\pi^d}V_T^{2d}V_T^6\int dc_1\int dc_2\, e^{-c_1^2}e^{-c_2^2}\left[\frac{2(d+2)(d-1)}{d}c_1^2c_2^4 - 2(d-1)c_2^6 - \frac{(d+2)^2(d-1)c_1^2c_2^2}{d} + c_2^4(d+2)(d-1)\right]$$

avec avant :

$$\frac{m}{2(d+2)}\left[V_1^2V_2^4\frac{2(d+2)(d-1)}{d} - 2(d-1)V_2^6 - \frac{d+2}{2}\frac{2}{d}(d+2)(d-1)V_T^2V_1^2V_2^2 + 2(d-1)\frac{d+2}{2}V_T^2V_2^4\right]$$

$$= \frac{m}{2(d+2)}\left[V_1^2V_2^4\frac{2(d+2)(d-1)}{d} - V_2^6\,2(d-1) - \frac{(d+2)^2(d-1)}{d}V_T^2V_1^2V_2^2 + (d-1)(d+2)V_T^2V_2^4\right] \checkmark$$

$\Rightarrow$

$$\dots = \frac{m}{2\cancel{(d+2)}}n^2V_T^6\left[\frac{2\cancel{(d+2)}(d-1)}{\cancel{d}}\frac{\cancel{d}}{2}\frac{d+2}{2} - \cancel{2}(d-1)\frac{d+4}{\cancel{2}}\frac{\cancel{d+2}}{2}\frac{d}{2} - \frac{(d+2)^{\cancel{2}}(d-1)}{\cancel{d}}\frac{\cancel{d}}{2}\frac{d}{2} + \cancel{(d+2)}(d-1)\frac{d+2}{2}\frac{d}{2}\right]$$

$$= \frac{m}{8}n^2V_T^6\left[2(d-1)(d+2) - (d-1)(d+4)d - d(d-1)\cancel{(d+2)} + d(d-1)\cancel{(d+2)}\right]$$

$$= \frac{m}{8}n^2V_T^6(d-1)\left[2d+4-d^2-4d\right]$$

$$= mn^2V_T^6\frac{d-1}{8}\left[-2d-d^2+4\right]$$

$$= -mn^2V_T^6\frac{(d-1)(d^2+2d-4)}{8} \quad : \boxed{OK}$$

Suite :

$$\frac{2m\beta^3}{d(d+2)nV_0}\int_{\mathbb{R}^d}dv\, S_i(\underline{v})L_c[MS_i] =$$

$$= -\sigma^{d-1}\frac{\phi V_T}{\cancel{\dots}}\frac{m}{2}\frac{\cancel{d}}{d}\frac{2m\beta^3}{d(d+2)\cancel{n}V_0}(-1)mn^{\cancel{2}}V_T^6\frac{(d-1)(d^2+2d-4)}{8}$$

$$= +\sigma^{d-1}\phi V_T\frac{\cancel{m^3}}{\cancel{2}}\frac{1}{d}\frac{\cancel{2}\beta^3}{d(d+2)V_0}n\frac{\cancel{8}}{\cancel{\beta^3 m^3}}\frac{(d-1)(d^2+2d-4)}{\cancel{8}}$$

$$= \frac{(d-1)(d^2+2d-4)}{d^2(d+2)}\frac{\sigma^{d-1}\phi V_T n}{V_0}$$

$$= \frac{(d-1)(d^2+2d-4)}{d^2\cancel{(d+2)}}\cancel{\sigma^{d-1}}\phi\cancel{V_T n}\frac{1}{\cancel{\dots}}\frac{\cancel{d+2}}{8}\frac{\Gamma(d/2)\sqrt{2}}{\cancel{\sqrt{2}}\,\pi^{(d-1)/2}}$$

$$= \frac{(d-1)(d^2+2d-4)\sqrt{2}}{8\cancel{\sqrt{2}}\,d^2}\phi\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}$$

Résumé

$$V_\eta^{*a} = \phi\frac{d+2}{8}\frac{\Gamma(d/2)\sqrt{2}}{\cancel{\sqrt{2}}\,\pi^{(d-1)/2}}$$

$$V_\eta^{*a'} = 0$$

$$V_\eta^{*c} = \phi\frac{1}{4}\frac{\Gamma(d/2)\sqrt{2}}{\cancel{\sqrt{2}}\,\pi^{(d-1)/2}}$$

$$\Rightarrow V_\eta^* = p\,\phi\frac{d+2}{8}\frac{\Gamma(d/2)}{\sqrt{2}\,\pi^{(d-1)/2}} + (1-p)\,\phi\frac{1}{4}\frac{\Gamma(d/2)}{\sqrt{2}\,\pi^{(d-1)/2}}$$

$$= \boxed{\phi\frac{1}{4}\frac{\Gamma(d/2)\sqrt{2}}{\cancel{\sqrt{2}}\,\pi^{(d-1)/2}}\left[p\frac{d+2}{2} + (1-p)\right]}$$

$$V_\kappa^{*a} = \phi\frac{d+2}{8}\frac{\Gamma(d/2)\sqrt{2}}{\cancel{\sqrt{2}}\,\pi^{(d-1)/2}}$$

$$V_\kappa^{*a'} = 0$$

$$V_\kappa^{*c} = \phi\frac{(d-1)(d^2+2d-4)\sqrt{2}}{8\cancel{\sqrt{2}}\,d^2}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}$$

$$\Rightarrow V_\kappa^* = p\,\phi\frac{d+2}{8}\frac{\Gamma(d/2)}{\sqrt{2}\,\pi^{(d-1)/2}} + (1-p)\,\phi\frac{(d-1)(d^2+2d-4)}{8\sqrt{2}\,d^2}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}$$

$$= \boxed{\phi\frac{1}{4}\frac{\Gamma(d/2)\sqrt{2}}{\cancel{\sqrt{2}}\,\pi^{(d-1)/2}}\left[p\frac{d+2}{2} + (1-p)\frac{(d-1)(d^2+2d-4)}{2d^2}\right]}$$

$$\begin{cases} \eta^* = 1/V_\eta^* \\ \kappa^* = \frac{d-1}{d}\frac{1}{V_\kappa^*} \\ \mu^* = 0 \end{cases}$$

Quel choix par $\varnothing$? Même choix que Santos : par analogie des Eq. de Boltzmann :

~~On obtient~~ $\cancel{\varnothing = V_0 \frac{d+2}{n\,2}}$ ~~(analogie avec Eq. Boltz)~~

$$\frac{\omega}{n\Omega_d} = \sigma^{d-1}\frac{\varnothing V_T}{S_d} \quad ; \; S_d = \Omega_d \quad ; \; \omega = V_0\frac{d+2}{2}$$

$$\Rightarrow \varnothing = \frac{\omega}{n\cancel{\Omega_d}}\frac{\cancel{S_d}}{V_T\sigma^{d-1}}$$

$$V_0 = nkT\frac{8}{d+2}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}\frac{\sigma^{d-1}}{\sqrt{mkT}} = n\sqrt{\frac{kT}{m}}\frac{8}{d+2}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}\sigma^{d-1}$$

$$= \frac{1}{nV_T\sigma^{d-1}}\frac{d+2}{2}V_0$$

$$= \frac{1}{\cancel{n}V_T\cancel{\sigma^{d-1}}}\frac{d+2}{2}\cancel{n\sigma^{d-1}}\sqrt{\frac{kT}{m}}\frac{4\,\Omega_d}{\pi^{1/2}(d+2)} \quad , \; \Omega_d = 2\pi^{d/2}/\Gamma(d/2)$$

$$= \sqrt{\frac{\beta m}{2}}\cancel{\frac{d+2}{2}}\sqrt{\frac{1}{m\beta}}\frac{4}{\pi^{1/2}\cancel{(d+2)}}\cancel{2}\pi^{d/2}\frac{1}{\Gamma(d/2)}$$

$$= \frac{4\pi^{d/2}}{\sqrt{2}\,\pi^{1/2}\,\Gamma(d/2)}$$

$$\boxed{\varnothing = \frac{4}{\sqrt{2}}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}}$$

On a donc :

$$V_\eta^* = \frac{\cancel{4}\,\sqrt{2}}{\cancel{\sqrt{2}}}\frac{\cancel{\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)}}\;\frac{1}{\cancel{4}}\frac{\cancel{\Gamma(d/2)}}{\sqrt{2}\,\cancel{\pi^{(d-1)/2}}}\left[\;\dots\;\right]$$

$\rho = 0$, $d = 3$: facteur 1/2 de trop par rapport aux simulations

$$\Rightarrow \boxed{\begin{aligned} V_\eta^* &= \cancel{\tfrac{1}{2}}\left[\rho\frac{d+2}{2} + (1-\rho)\right] \\ V_\kappa^* &= \cancel{\tfrac{1}{2}}\left[\rho\frac{d+2}{2} + (1-\rho)\frac{(d-1)(d^2+2d-4)}{2d^2}\right]\end{aligned}}$$

$$\boxed{\begin{aligned}\eta^* &= \frac{1}{V_\eta^*} \\ \kappa^* &= \frac{d-1}{d}\frac{1}{V_\kappa^*} \\ \mu^* &= 0\end{aligned}}$$

~~$\rho = 0$, $d=3$: $2\frac{2}{2}\frac{2\cdot 9}{2(9+6-4)} = 2\frac{6}{11} = \frac{12}{11}$~~

~~$\Rightarrow$ surestimation ~ 40%~~

$d=3$: $\frac{2(9+6-4)}{2\cdot 9} = \frac{11}{9}$

La distribution à l'ordre 1 est alors (4.67) :

$$\boxed{f^{(1)}(V) = -\frac{\beta^3}{n}\mathcal{M}(\underline{V})\left[\frac{2m}{d+2}S_i(\underline{V})\,\kappa\,\nabla_i T + \frac{\eta}{\beta}D_{ij}(V)\nabla_j u_i\right]}$$

Equations hydrodynamiques : calcul similaire, mais avec $\xi_T^{(1)} = \mu^* = 0$: $\xi_{u_i}^{(1)} = 0$

1) $\left[\partial_\tau + 2\rho n\xi_n^{(0)*}\right]\rho_{\underline{k}}(\tau) + \rho\,\xi_n^{(0)*}\theta_{\underline{k}}(\tau) + ik\,W_{\underline{k}_\parallel}(\tau) = 0$

2) $\left[\partial_\tau + \frac{d-1}{d}\eta^* k^2\right]\underline{W}_{k_\parallel} + i\underline{k}\left[\;\rho_{\underline{k}}(\tau) + (1-\rho\,\xi_n^* k^*)\theta_{\underline{k}}(\tau)\right] = 0$

3) $\left[\partial_\tau + \frac{1}{2}\eta^* k^2\right]\underline{W}_{k_\perp}(\tau) = 0$

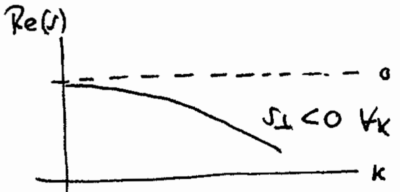
4) $\left[\partial_\tau + \frac{d+2}{2(d-1)}\kappa^* k^2\right]\theta_{\underline{k}}(\tau) + \frac{2}{d}\,ik\,W_{k_\parallel}(\tau) = 0$

De 3) on en tire le mode perpendiculaire :

car $\xi_T^{(1)} = 0$

$$W_{k_\perp}(\tau) = W_{k_\perp}(0)\exp\left[S_\perp(\rho,k)\,\tau\right] \quad ; \quad S_\perp(\rho,k) = -\frac{1}{2}\eta^* k^2 \quad < 0 \quad \forall \rho, k$$

Conclusion :



$\Rightarrow \nexists$ instabilités pour le modèle de Maxwell !
(modulo ~~le couplage~~ les autres modes...) linéaires

Calcul des taux de déclin au premier ordre : reprend résultats précédents :

$$\omega\left[Af^{(1)}, Bf^{(1)}\right] = -\sigma^{d-1}\varnothing V_T\frac{\beta^3 n}{\pi^d}\frac{2m}{d+2}\left(\kappa\nabla_i T + \mu\nabla_i n\right)\int_{\mathbb{R}^{2d}} dc_1 dc_2\,A(V_T c_2)\,B(V_T c_1)\,e^{-c_1^2}e^{-c_2^2}\left(\frac{m}{2}V_T^2 c_1^2 - \frac{d+2}{2}k_B T\right)V_T c_{1i}$$

$$-\sigma^{d-1}\varnothing V_T\frac{\beta^3 n}{\pi^d}\;\frac{\eta}{\beta}\nabla_j u_i\int_{\mathbb{R}^{2d}} dc_1 dc_2\,A(V_T c_2)\,B(V_T c_1)\,e^{-c_1^2}e^{-c_2^2}\,m\left(V_T^2 c_{1i}c_{1j} - \frac{1}{d}V_T^2 c_1^2\delta_{ij}\right)$$

$$= -\sigma^{d-1}\phi v_T \frac{\beta^3 n}{\pi^d}\frac{2m}{d+2}\kappa\nabla_i T\int_{\mathbb{R}^{2d}} dc_1 dc_2\, A(v_T c_2)B(v_T c_1)e^{-c_1^2}e^{-c_2^2}\frac{m}{2}v_T^2\left(c_1^2-\frac{d+2}{2}\right)v_T c_{1i}$$
$$-\sigma^{d-1}\phi v_T\frac{\beta^3 n}{\pi^d}\frac{\eta}{\beta}\nabla_j u_i\int_{\mathbb{R}^{2d}} dc_1 dc_2\, A(v_T c_2)B(v_T c_1)e^{-c_1^2}e^{-c_2^2} m v_T^2\left(c_{1i}c_{1j}-\frac{1}{d}c_1^2\delta_{ij}\right)$$

$$= -\sigma^{d-1}\phi v_T\frac{\beta^3 n}{\pi^d}\frac{2m}{d+2}\kappa\nabla_i T\frac{m}{2}v_T^2 v_T\, I_1 - \sigma^{d-1}\phi v_T\frac{\beta^3 n}{\pi^d}\frac{\eta}{\beta}\nabla_j u_i m v_T^2\, I_2$$

où :

$$I_1 = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, A(v_T c_2)B(v_T c_1)e^{-c_1^2}e^{-c_2^2}\left(c_1^2-\frac{d+2}{2}\right)c_{1i}$$
$$I_2 = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, A(v_T c_2)B(v_T c_1)e^{-c_1^2}e^{-c_2^2}\left(c_{1i}c_{1j}-\frac{1}{d}c_1^2\delta_{ij}\right)$$

Simplification de $\omega[Af^{(0)},Bf^{(1)}]$:

$$\omega[Af^{(0)},Bf^{(1)}] = -\sigma^{d-1}\phi v_T^4\frac{\beta^3 n}{\pi^d}\frac{\cancel{2}m^2}{d+2}\frac{\cancel{m}}{\cancel{2}}T\kappa_0\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1 - \sigma^{d-1}\phi v_T^3\frac{\beta^{\cancel{3}\,2}n}{\pi^d}m\frac{\cancel{\eta}}{\cancel{\beta}}\eta_0\left(\eta^*\nabla_i u_j\right)$$

$$= -\cancel{\sigma^{d-1}}\phi\frac{4}{\cancel{\beta^2}}\frac{\cancel{\beta^3}n}{\pi^d}\frac{\cancel{m^2}}{\cancel{d+2}}\cancel{T}\frac{d\cancel{(d+2)}}{2(d-1)}\frac{\cancel{k_B}}{m}\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{\sqrt{mk_BT}}{\cancel{\sigma^{d-1}}}\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1$$

$$-\cancel{\sigma^{d-1}}\phi\frac{2}{\cancel{\beta}m}v_T\frac{\beta n}{\pi^d}m\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{\sqrt{mk_BT}}{\cancel{\sigma^{d-1}}}\left(\eta^*\nabla_i u_j\right)I_2$$

$$= -\phi\, 4\frac{n}{\pi^d}\frac{d+2}{16m(d-1)}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\sqrt{\frac{2m}{\beta}}\frac{1}{\sqrt{2}}\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1 - \phi\frac{2}{\cancel{\beta}}\sqrt{\frac{2}{\cancel{\beta m}}}\frac{\cancel{\beta}n}{\pi^d}\cancel{m}\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\cancel{\sqrt{m/\beta}}\left(\eta^*\nabla_i u_j\right)I_2$$

$$= -\phi\, 4\frac{n}{\pi^d}\frac{d(d+2)}{16(d-1)}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{1}{\sqrt{2}}v_T\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1 - \phi\, 2\sqrt{2}\frac{n}{\pi^d}\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\left(\eta^*\nabla_i u_j\right)I_2 \quad \checkmark$$

$$= -\frac{\cancel{4}}{\sqrt{2}}\frac{\cancel{\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)}}\frac{n}{\pi^d}\frac{d(d+2)}{\cancel{4}(d-1)}\frac{\cancel{\Gamma(d/2)}}{\cancel{\pi^{(d-1)/2}}}\frac{1}{\sqrt{2}}v_T\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1 - \frac{\cancel{8}}{\cancel{\sqrt{2}}}\frac{\cancel{\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)}}\cancel{2\sqrt{2}}\frac{n}{\pi^d}\frac{d+2}{\cancel{8}}\frac{\cancel{\Gamma(d/2)}}{\cancel{\pi^{(d-1)/2}}}\left(\eta^*\nabla_i u_j\right)I_2$$

$$= -\frac{d(d+2)}{2(d-1)}v_T\frac{n}{\pi^d}\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1 - (d+2)\frac{n}{\pi^d}\left(\eta^*\nabla_i u_j\right)I_2$$

<u>Calcul des différents termes</u> :

$$\xi_n^{(1)} = \frac{2}{n}\omega[f^{(0)},f^{(1)}]$$
$$= -\frac{2}{n}\left[\frac{d+2}{2(d-1)}\frac{v_T n}{\pi^d}\left(\kappa^*\frac{1}{T}\nabla_i T\right)I_1^{1,1} + (d+2)\frac{n}{\pi^d}\left(\eta^*\nabla_i u_j\right)I_2^{1,1}\right]$$

$$I_1^{1,1} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left[c_1^2 c_{1i} - \frac{d+2}{2}c_{1i}\right]$$
$$= \underbrace{\left(\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}c_1^2 c_{1i}\right)}_{=0}\underbrace{\left(\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}\right)}_{=\pi^{d/2}} - \frac{d+2}{2}\underbrace{\left(\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}c_{1i}\right)}_{=0}\underbrace{\left(\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}\right)}_{=\pi^{d/2}}$$
$$= 0$$

$$I_2^{1,1} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left[c_{1i}c_{1j} - \frac{1}{d}c_1^2\delta_{ij}\right]$$
$$= \underbrace{\left(\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}c_{1i}c_{1j}\right)}_{=\pi^{d/2}\frac{d}{2d}\delta_{ij}}\underbrace{\left(\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}\right)}_{=\pi^{d/2}} - \frac{1}{d}\underbrace{\left(\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}c_1^2\right)}_{=\pi^{d/2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}}\underbrace{\left(\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2}\right)}_{=\pi^{d/2}}\delta_{ij}$$

$$= \pi^d\frac{1}{2}\delta_{ij} - \pi^d\delta_{ij}\frac{1}{d}\frac{d}{2}$$

$$= 0$$

$$\Rightarrow \boxed{\xi_n^{(1)} = 0}$$

$$\xi_T^{(1)} = -\underbrace{\xi_n^{(1)}}_{=0} + \frac{m}{nk_BTd}\omega[f^{(0)},V^2 f^{(1)}] + \frac{m}{nk_BTd}\omega[V^2 f^{(0)},f^{(1)}]$$

$$I_1^{1,V_1^2} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, V_T^2 c_1^2 e^{-c_1^2} e^{-c_2^2} \left(c_1^2 - \frac{d+2}{2}\right) c_{1i}$$

$$= V_T^2 \pi^{d/2} \left[\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c^4 c_{1i}}_{=0} - \frac{d+2}{2} \underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c^2 c_{1i}}_{=0} \right] = 0$$

$$I_1^{V_2^2,1} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, V_T^2 c_2^2 e^{-c_1^2} e^{-c_2^2} \left(c_1^2 - \frac{d+2}{2}\right) c_{1i} = 0$$

$$I_2^{1,V_1^2} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, V_T^2 c_1^2 e^{-c_1^2} e^{-c_2^2} \left(c_{1i} c_{1j} - \frac{1}{d} c_1^2 \delta_{ij}\right)$$

$$= V_T^2 \pi^{d/2} \left[\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c^2 c_i c_j}_{=\pi^{d/2} \frac{d+2}{2d} \frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)} \delta_{ij}} - \frac{1}{d} \delta_{ij} \underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c^4}_{=\pi^{d/2} \frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}} \right]$$

$$= V_T^2 \pi^d \delta_{ij} \left[\frac{d+2}{2\cancel{d}} \frac{\cancel{d}}{2} - \frac{1}{\cancel{d}} \frac{d+2}{2} \frac{\cancel{d}}{2} \right]$$

$$= 0$$

$$I_2^{V_2^2,1} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, V_T^2 c_2^2 e^{-c_1^2} e^{-c_2^2} \left(c_{1i} c_{1j} - \frac{1}{d} c_1^2 \delta_{ij}\right)$$

$$= V_T^2 \Bigg(\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c^2}_{=\pi^{d/2} \frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}} \Bigg) \left[\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c_{1i} c_{1j}}_{=\pi^{d/2} \frac{d}{2d} \delta_{ij}} - \frac{1}{d} \delta_{ij} \underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c^2}_{=\pi^{d/2} \frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}} \right]$$

$$= V_T^2 \pi^{d/2} \frac{d}{2} \pi^{d/2} \delta_{ij} \left(\frac{1}{2} - \frac{1}{d} \frac{d}{2} \right)$$

$$= 0$$

$$\Rightarrow \boxed{\zeta_T^{(1)} = 0}$$

$$\zeta_{u_i}^{(1)} = \frac{1}{nV_T} \omega\left[f^{(0)}, V_i f^{(1)}\right] + \frac{1}{nV_T} \omega\left[V_i f^{(0)}, f^{(1)}\right]$$

$$I_1^{1,V_{1i}} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} c_{1i} V_T \left(c_1^2 - \frac{d+2}{2}\right) c_{1i}$$

$$= V_T \pi^{d/2} \left[\underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c_1^4}_{=\pi^{d/2} \frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}} - \frac{d+2}{2} \underbrace{\int_{\mathbb{R}^d} dc\, e^{-c^2} c_1^2}_{=\pi^{d/2} \frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}} \right]$$

$$= V_T \pi^d \left[\frac{d+2}{2} \frac{d}{2} - \frac{d+2}{2} \frac{d}{2} \right]$$

$$= 0$$

$$I_1^{V_{2i},1} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} V_T c_{2i} \left(c_1^2 - \frac{d+2}{2}\right) c_{1i} \overset{\text{symétries}}{=} 0$$

$$I_2^{1,V_{1i}} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} V_T c_{1i} \left(c_{1i} c_{1j} - \frac{1}{d} c_1^2 \delta_{ij}\right) \overset{\text{symétries}}{=} 0$$

$$I_2^{V_{2i},1} = \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} V_T c_{2i} \left(c_{1i} c_{1j} - \frac{1}{d} c_1^2 \delta_{ij}\right) \overset{\text{sym.}}{=} 0$$

$$\Rightarrow \boxed{\zeta_{u_i}^{(1)} = 0}$$

Conclusion : tous les taux de déclin à l'ordre 1 sont nuls ! Les éq. hydro. linéarisées sont alors:

$$\begin{cases} \left[\partial_\tau + 2p n \zeta_n^{(0)*}\right] \rho_k(\tau) + p\, \zeta_n^{(0)*} \theta_k(\tau) + i k w_{k\parallel}(\tau) = 0 \\ \left[\partial_\tau + \frac{d-1}{d} \eta^* k^2\right] w_{k\parallel}(\tau) + i k \left[\rho_k(\tau) + \theta_k(\tau)\right] = 0 \\ \left[\partial_\tau + \frac{1}{2} \eta^* k^2\right] w_{k\perp}(\tau) = 0 \\ \left[\partial_\tau + \frac{d+2}{2(d-1)} \kappa^* k^2\right] \theta_k(\tau) + \frac{2}{d} i k w_{k\parallel}(\tau) = 0 \end{cases}$$

; pour $k \to 0$: solution: $\rho(t) = \left[\rho(0) + \frac{\theta_k}{2}\right] \exp\left(-2p\zeta_n^{(0)*} \tau\right)$

$\Rightarrow$ toujours stable : on a donc les relations de dispersion pour $k \sim 0$: toujours stabilité avec Maxwell !!!

Coefficient de viscosité η :

Annihilation :

$$\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v}) L_a[M D_{ij}] \overset{\text{lemme 3.3.}}{=} \sigma^{d-1}\phi V_T \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1) M(\underline{v}_2) D_{ij}(\underline{v}_2)\left[D_{ij}(\underline{v}_1) + D_{ij}(\underline{v}_2)\right]$$

$$= \sigma^{d-1}\phi V_T \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2 \frac{n}{V_T^d}\frac{1}{\pi^{d/2}} e^{-V_1^2/V_T^2} \frac{n}{V_T^d}\frac{1}{\pi^{d/2}} e^{-V_2^2/V_T^2} m^2 \left(V_{2i}V_{2j} - \frac{V_2^2}{d}\delta_{ij}\right)\left(V_{2i}V_{2j} - \frac{V_2^2}{d}\delta_{ij} + V_{1i}V_{1j} - \frac{V_1^2}{d}\delta_{ij}\right)$$

$$= \frac{\sigma^{d-1}\phi V_T^5 n^2}{V_T^{2d}\,\pi^d} m^2 V_T^{2d} \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2}\Big(c_{2i}c_{2j}c_{2i}c_{2j} - \frac{V_2^2}{d}\delta_{ij}V_{2i}V_{2j} + V_{2i}V_{2j}V_{1i}V_{1j} - V_{2i}V_{2j}\frac{V_1^2}{d}\delta_{ij} \quad (V = c)$$
$$- \frac{V_2^2}{d}\delta_{ij}V_{2i}V_{2j} + \frac{V_2^4}{d^2}\delta_{ij}\delta_{ij} - \frac{V_2^2}{d}\delta_{ij}V_{1i}V_{1j} + \frac{V_1^2V_2^2}{d^2}\delta_{ij}\delta_{ij}\Big)$$

$$= \frac{\sigma^{d-1}\phi V_T^5 n^2 m^2}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left(c_2^4 - \frac{1}{d}c_2^4 + (c_1\cdot c_2)^2 - \frac{1}{d}c_1^2c_2^2 - \cancel{\frac{1}{d}c_2^4} + \cancel{\frac{1}{d}c_2^4} - \cancel{\frac{1}{d}c_1^2c_2^2} + \cancel{\frac{1}{d}c_1^2c_2^2}\right)$$

$$= \frac{\sigma^{d-1}\phi V_T^5 n^2 m^2}{\pi^d}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left[\frac{d-1}{d}c_2^4 + \cancel{\frac{1}{d}c_1^2}c_2^2 - \cancel{\frac{1}{d}c_1^2}c_2^2\right]$$

$$= \frac{\sigma^{d-1}\phi V_T^5 n^2 m^2}{\pi^d}\frac{d-1}{d}\int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}\int_{\mathbb{R}^d} dc_2\, e^{-c_2^2} c_2^4$$

$$= \sigma^{d-1}\phi V_T^5 n^2 m^2 \frac{d-1}{d}\frac{\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d/2)}$$

$$= \sigma^{d-1}\phi V_T^5 n^2 m^2 \frac{d-1}{d}\frac{d+2}{2}\frac{d}{2}$$

$$\nu_\eta^{*a} = \frac{\beta^2}{(d+2)(d-1)nV_0}\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v}) L_a[M D_{ij}]$$

$$= \frac{\beta^2}{\cancel{(d+2)}\cancel{(d-1)}nV_0}\sigma^{d-1}\phi V_T^5 n^2 m^2 \cancel{(d-1)}\frac{\cancel{d+2}}{2}\frac{1}{2}$$

$$= \frac{1}{\cancel{k_B^2T^2}}\frac{1}{\cancel{\phi}}\frac{1}{\cancel{k_BT}}\frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\frac{\cancel{\sqrt{mk_BT}}}{\cancel{\sigma^{d-1}}}\cancel{\sigma^{d-1}}\phi\sqrt{\frac{2\cancel{k_BT}}{\cancel{m}}}\frac{\cancel{k_B^2T^2}}{\cancel{m^2}}\frac{1}{\cancel{n}}\cancel{n^2}\cancel{m^2}$$

$$= \frac{d+2}{8}\frac{\Gamma(d/2)}{\pi^{(d-1)/2}}\phi\sqrt{2} \qquad ; \ \phi = \frac{4}{\sqrt{2}}\frac{\pi^{(d-1)/2}}{\Gamma(d/2)}$$

$$= \frac{d+2}{8}\frac{\cancel{\Gamma(d/2)}}{\cancel{\pi^{(d-1)/2}}}\cancel{\sqrt{2}}\frac{4}{\cancel{\sqrt{2}}}\frac{\cancel{\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)}}$$

$$= \frac{d+2}{2}$$

Collision :

$$\int_{\mathbb{R}^d} d\underline{v}\, D_{ij}(\underline{v}) L_c[M D_{ij}] \overset{\text{lemme 3.4}}{=} -\sigma^{d-1}\frac{\phi V_T}{S_d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1) M(\underline{v}_2) X(\underline{v}_2)\int d\hat{\sigma}\,(b-1)\left[Y(v_1) + Y(v_2)\right] \quad ; X(v_2) = D_{ij}(v_2) ; Y(v_1) = D_{ij}(v_1)$$

$$= -\sigma^{d-1}\frac{\phi V_T}{S_d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2 \frac{n^2}{V_T^{d2}}\frac{1}{(\pi^d)^{1/2}} e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2} D_{ij}(v_2)\underbrace{\int d\hat{\sigma}\,(b-1)\left[D_{ij}(v_1) + D_{ij}(v_2)\right]}\, m$$

$$\overset{\text{notes}}{=} \int d\hat{\sigma}\,(g\cdot\hat{\sigma})\left[-g_i\sigma_j - g_j\sigma_i + 2(g\cdot\hat{\sigma})\sigma_i\sigma_j\right]$$

$$= -\sigma^{d-1}\frac{\phi V_T}{S_d}\frac{n^2}{V_T^{2d}}\frac{1}{(\pi^d)^{1/2}} m \int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2} D_{ij}(v_2)\int d\hat{\sigma}\,(g\cdot\hat{\sigma})\left[-g_i\sigma_j - g_j\sigma_i + 2(g\cdot\hat{\sigma})\sigma_i\sigma_j\right]$$

Avec :

$$-\int d\hat{\sigma}\,(g\cdot\hat{\sigma})g_i\sigma_j = -g_i\int d\hat{\sigma}\,(g\cdot\hat{\sigma})\sigma_j = -g_i\beta_2 g_j \qquad ; \beta_n = 2\pi^{(d-1)/2}\,\Gamma\left(\tfrac{n+1}{2}\right)/\Gamma\left(\tfrac{n+d}{2}\right)$$

$$-\int d\hat{\sigma}\,(g\cdot\hat{\sigma})g_j\sigma_i = -g_j\int d\hat{\sigma}\,(g\cdot\hat{\sigma})\sigma_i = -g_j\beta_2 g_i$$

$$2\int d\hat{\sigma}\,(g\cdot\hat{\sigma})^2\sigma_i\sigma_j = 2\frac{\beta_2}{2+d}\left(2g_ig_j + g^2\delta_{ij}\right)$$

Il vient :

$\boxed{B}$

$$\int_{\mathbb{R}^d} dv\, D_{ij}(v) L_c[M D_{ij}] = -\sigma^{d-1}\frac{\phi V_T}{S_d}\frac{n^2}{V_T^{2d}}\frac{m^2}{(\pi^{d/2})^2}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2}\left(V_{2i}V_{2j}-\frac{V_2^2}{d}\delta_{ij}\right)\left[-2g_ig_j\beta_2+\frac{2\beta_2}{d+2}(2g_ig_j+g^2\delta_{ij})\right]$$

$$= -\sigma^{d-1}\frac{\phi V_T}{S_d}\frac{n^2}{V_T^{2d}}\frac{m^2}{(\pi^{d/2})^2}2\beta_2\int_{\mathbb{R}^{2d}} dv_1 dv_2\, e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2}\left(V_{2i}V_{2j}-\frac{V_2^2}{d}\delta_{ij}\right)\underbrace{\left[-g_ig_j+\frac{2}{d+2}g_ig_j+\frac{1}{d+2}g^2\delta_{ij}\right]}_{=g_ig_j\frac{\cancel{2}-d-\cancel{2}}{d+2}+g^2\delta_{ij}\frac{1}{d+2}}$$

$$= -\sigma^{d-1}\frac{\phi V_T}{S_d}\frac{n^2}{V_T^{2d}}\frac{m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2}\left(V_{2i}V_{2j}-\frac{V_2^2}{d}\delta_{ij}\right)\underbrace{\left[g^2\delta_{ij}-d\,g_ig_j\right]}_{\substack{=\delta_{ij}(V_1^2+V_2^2-2V_{1k}V_{2k})\\ -d(V_1-V_2)_i(V_1-V_2)_j}}$$

$$= -\sigma^{d-1}\frac{\phi V_T}{S_d}\frac{n^2}{V_T^{2d}}\frac{m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dv_1 dv_2\, e^{-V_1^2/V_T^2} e^{-V_2^2/V_T^2}\left(V_{2i}V_{2j}-\frac{V_2^2}{d}\delta_{ij}\right)\Big(V_1^2\delta_{ij}+V_2^2\delta_{ij}-2V_{1k}V_{2k}\delta_{ij}$$
$$-d\,V_{1i}V_{1j}+d\,V_{1i}V_{2j}$$
$$+d\,V_{2i}V_{1j}-d\,V_{2i}V_{2j}\Big)$$

$$= -\sigma^{d-1}\frac{\phi V_T}{S_d}\frac{n^2}{\cancel{V_T^{2d}}}\frac{m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}\cancel{V_T^{2d}}\,V_T^4\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left(c_{2i}c_{2j}-\frac{c_2^2}{d}\delta_{ij}\right)\Big(c_1^2\delta_{ij}+c_2^2\delta_{ij}-2\cancel{c_{1k}c_{2k}}^{\,0}\delta_{ij}$$
$$-d\,c_{1i}c_{1j}+d\,\cancel{c_{1i}}^{\,0}c_{2j}$$
$$+d\,\cancel{c_{2i}}c_{1j}^{\,0}-d\,c_{2i}c_{2j}\Big)$$

$$= -\sigma^{d-1}\frac{\phi V_T^5}{S_d}\frac{n^2m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\Big(c_{2i}c_{2j}\,c_1^2\delta_{ij}+c_{2i}c_{2j}\,c_2^2\delta_{ij}-d\,c_{2i}c_{2j}c_{1i}c_{1j}$$
$$-d\,c_{2i}c_{2j}c_{2i}c_{2j}-\frac{1}{d}c_2^2\delta_{ij}c_1^2\delta_{ij}-\frac{1}{d}c_2^2\delta_{ij}c_2^2\delta_{ij}$$
$$+\frac{1}{d}c_2^2\delta_{ij}\,d\,c_{1i}c_{1j}+\frac{1}{d}c_2^2\delta_{ij}\,d\,c_{2i}c_{2j}\Big)$$

$$= -\sigma^{d-1}\frac{\phi V_T^5}{S_d}\frac{n^2m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\Big(c_1^2c_2^2+c_2^4-d\,(c_1\cdot c_2)^2-d\,c_2^4-\cancel{\frac{1}{d}d\,c_1^2c_2^2}-\cancel{\frac{1}{d}d\,c_2^4}$$
$$+\cancel{c_2^2c_1^2}+\cancel{c_2^4}\Big)$$

$$= -\sigma^{d-1}\frac{\phi V_T^5}{S_d}\frac{n^2m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}\left[\cancel{c_1^2c_2^2}+c_2^4(1-d)-d\frac{1}{d}\cancel{c_1^2c_2^2}\right]$$

$$= \sigma^{d-1}\frac{\phi V_T^5}{S_d}\frac{n^2m^2}{(\pi^{d/2})^2}\frac{2\beta_2}{d+2}(d-1)\,\pi^{d/2}\int_{\mathbb{R}^d} dc\, e^{-c^2}c^4$$

$$= \sigma^{d-1}\frac{\phi V_T^5}{S_d}\frac{n^2m^2}{\pi^{d/2}}\frac{2\beta_2}{d+2}(d-1)\,\pi^{d/2}\frac{\Gamma\left(\frac{d+4}{2}\right)}{\Gamma(d/2)}$$

$$= \sigma^{d-1}\frac{\phi V_T^5}{S_d}n^2m^2\frac{2\beta_2}{\cancel{d+2}}(d-1)\frac{\cancel{d+2}}{2}\frac{d}{2}$$

$$\nu_\eta^{*c} = \frac{\beta^2}{(d+2)(d-1)n\nu_0}\int_{\mathbb{R}^d} dv\, D_{ij}(v)L_c[MD_{ij}]$$

$$= \frac{\beta^2}{(d+2)(d-1)n\nu_0}\frac{\sigma^{d-1}\phi V_T^4}{S_d}\sqrt{\frac{2k_BT}{m}}\,n^2m^2\,2\beta_2(d-1)\frac{d}{4}$$

$$= \frac{1}{\cancel{k_BT^2}}\frac{1}{(d+2)\cancel{(d-1)}}\frac{1}{\cancel{n}}\frac{1}{\cancel{k_BT}}\frac{d+2}{8}\frac{\Gamma(d/2)}{\cancel{\pi^{(d-1)/2}}}\frac{\sqrt{\pi}\cancel{k_BT}}{\cancel{\sigma^{d-1}}}\cancel{\sigma^{d-1}}\frac{1}{2\pi^{d/2}/\cancel{\Gamma(d/2)}}\frac{4}{\sqrt{2}}\frac{\cancel{\pi^{(d-1)/2}}}{\cancel{\Gamma(d/2)}}\frac{\cancel{4k_BT^2}}{\cancel{m^2}}\sqrt{\frac{2k_BT}{\cancel{m}}}\cancel{n^2}\cancel{m^2}\,2\,\cancel{(d-1)}\frac{d}{\cancel{4}}2\pi^{(d-1)/2}\frac{\Gamma\left(\frac{1+2}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)}$$

$$= \frac{1}{\cancel{d+2}}\frac{\cancel{d+2}}{\cancel{8}}\cancel{\Gamma(d/2)}\frac{1}{\cancel{2\pi^{d/2}}}\frac{\cancel{4}}{\cancel{\sqrt{2}}}\cancel{\sqrt{2}}\,\cancel{2d}\,\cancel{2\pi^{d/2}}\,\cancel{\pi^{-1/2}}\,\cancel{\tfrac{1}{2}}\cancel{\sqrt{\pi}}\frac{1}{\cancel{\frac{d}{2}}\cancel{\Gamma(d/2)}}$$

$$= 1$$

<u>Conclusion</u>:

$$\nu_\eta^* = p\,\frac{d+2}{2}+(1-p) \quad \Longrightarrow \quad \boxed{\eta^* = \frac{1}{\nu_\eta^*} = \frac{1}{p\,\frac{d+2}{2}+(1-p)}}$$

Coefficient de conductivité thermique :

$\boxed{C}$

Collision :

$$\int_{\mathbb{R}^d} dv\, S_i(\underline{v})\, L_c[MS_i] \overset{\text{lemma 3.4.}}{=} -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\, M(\underline{v}_2)\, S_i(\underline{v}_2)\int d\hat{\sigma}\,(b-1)\left[S_i(v_1)+S_i(v_2)\right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, f^{(0)}(\underline{v}_1)\, M(\underline{v}_2)\, S_i(\underline{v}_2)\int d\hat{\sigma}(b-1)\left[\left(\frac{m}{2}\,V_1^2-\frac{d+2}{2}k_BT\right)V_{1i}+\left(\frac{m}{2}\,V_2^2-\frac{d+2}{2}k_BT\right)V_{2i}\right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\frac{m}{2}\int_{\mathbb{R}^{2d}} d\underline{v}_1 d\underline{v}_2\, \frac{n^2}{V_T^{2d}}\frac{1}{(\pi^{d/2})^2}e^{-V_1^2/V_T^2}e^{-V_2^2/V_T^2}S_i(v_2)\int d\hat{\sigma}(b-1)\left[\left(V_1^2-\frac{d+2}{2}\frac{2}{m\beta}\right)V_{1i}+\left(V_2^2-\frac{d+2}{2}\frac{2}{m\beta}\right)V_{2i}\right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T}{S_d}\frac{m^2}{2^2}\frac{n^2}{\cancel{V_T^{2d}}}\frac{1}{\pi^d}\cancel{V_T^{2d}}\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2}e^{-c_2^2}c_{2i}\left(c_2^2-\frac{d+2}{2}\right)V_T^3\int d\hat{\sigma}(b-1)\left[V_T^3\left(c_1^2-\frac{d+2}{2}\right)c_{1i}+V_T^3\left(c_2^2-\frac{d+2}{2}\right)c_{2i}\right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T^{6+1}}{S_d}\frac{m^2n^2}{4\pi^d}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2}c_{2i}\left(c_2^2-\frac{d+2}{2}\right)\int d\hat{\sigma}(b-1)\left[\left(c_1^2-\frac{d+2}{2}\right)c_{1i}+\left(c_2^2-\frac{d+2}{2}\right)c_{2i}\right]$$

$$= -\sigma^{d-1}\frac{\emptyset V_T^{7}}{S_d}\frac{m^2n^2}{4\pi^d}\int_{\mathbb{R}^{2d}} dc_1dc_2\, e^{-c_1^2}e^{-c_2^2}\left(c_2^2-\frac{d+2}{2}\right)c_{2i}\int d\hat{\sigma}(b-1)\left[c_1^2c_{1i}+c_2^2c_{2i}-\frac{d+2}{2}(c_{1i}+c_{2i})\right]$$

Avec :

$$(b-1)(c_{1i}+c_{2i}) = \cancel{c_{1i}}-\cancel{(c_1\cdot\sigma)\sigma_i}+\cancel{c_{2i}}+\cancel{(c_2\cdot\sigma)\sigma_i}-\cancel{c_{1i}}-\cancel{c_{2i}} = 0$$

$$(b-1)\left[c_1^2c_{1i}+c_2^2c_{2i}\right] = \left[c_1-(g\cdot\hat{\sigma})\hat{\sigma}\right]^2\left[c_{1i}-(g\cdot\hat{\sigma})\sigma_i\right]+\left[c_2+(g\cdot\hat{\sigma})\hat{\sigma}\right]^2\left[c_{2i}+(g\cdot\hat{\sigma})\sigma_i\right]-\dots$$

$$= \left[c_1^2+(g\cdot\hat{\sigma})^2-2(g\cdot\hat{\sigma})\sigma_jc_{1j}\right]\left[c_{1i}-(g\cdot\hat{\sigma})\sigma_i\right]+\left[c_2^2+(g\cdot\hat{\sigma})^2+2(g\cdot\hat{\sigma})\sigma_jc_{2j}\right]\left[c_{2i}+(g\cdot\hat{\sigma})\sigma_i\right]-\dots$$

$$= \left[\cancel{c_1^2c_{1i}}-(g\cdot\hat{\sigma})c_1^2\sigma_i+(g\cdot\hat{\sigma})^2c_{1i}-\cancel{(g\cdot\hat{\sigma})^3\sigma_i}-2(g\cdot\hat{\sigma})\sigma_jc_{1j}c_{1i}+2(g\cdot\hat{\sigma})^2\sigma_jc_{1j}\sigma_i\right]$$
$$+\left[\cancel{c_2^2c_{2i}}+(g\cdot\hat{\sigma})c_2^2\sigma_i+(g\cdot\hat{\sigma})^2c_{2i}+\cancel{(g\cdot\hat{\sigma})^3\sigma_i}+2(g\cdot\hat{\sigma})\sigma_jc_{2j}c_{2i}+2(g\cdot\hat{\sigma})^2\sigma_jc_{2j}\sigma_i\right]$$
$$-\cancel{c_1^2c_{1i}}-\cancel{c_2^2c_{2i}}$$

$$= (g\cdot\hat{\sigma})^2\left[c_{1i}+c_{2i}+2\sigma_i\sigma_jc_{1j}+2\sigma_i\sigma_jc_{2j}\right]$$
$$+(g\cdot\hat{\sigma})\left[-c_1^2\sigma_i+c_2^2\sigma_i-2\sigma_jc_{1j}c_{1i}+2\sigma_jc_{2j}c_{2i}\right]$$

$$= (g\cdot\hat{\sigma})^2(c_{1j}+c_{2j})\left[\delta_{ij}+2\sigma_i\sigma_j\right]-(g\cdot\hat{\sigma})\sigma_j\left[c_1^2\delta_{ij}-c_2^2\delta_{ij}+2c_{1i}c_{1j}-2c_{2i}c_{2j}\right]$$

$$\int d\hat{\sigma}(g\cdot\hat{\sigma})^2(c_{1j}+c_{2j})\delta_{ij} = (c_{1j}+c_{2j})\beta_2\,g^2\delta_{ij}\qquad;\ \beta_n = 2\pi^{(d-1)/2}\,\Gamma\!\left(\tfrac{n+1}{2}\right)\Big/\Gamma\!\left(\tfrac{n+d}{2}\right)$$

$$\int d\hat{\sigma}(g\cdot\hat{\sigma})^2(c_{1j}+c_{2j})2\sigma_i\sigma_j = 2(c_{1j}+c_{2j})\frac{\beta_2}{d+2}\left(2g_ig_j+g^2\delta_{ij}\right)$$

$$-\int d\hat{\sigma}(g\cdot\hat{\sigma})\left(c_1^2\delta_{ij}-c_2^2\delta_{ij}+2c_{1i}c_{1j}-2c_{2i}c_{2j}\right)\sigma_j = -\left(c_1^2\delta_{ij}-c_2^2\delta_{ij}+2c_{1i}c_{1j}-2c_{2i}c_{2j}\right)\beta_2\,g_j$$

Ainsi :

$$c_{2i}(c_{1i}+c_{2i})\beta_2\left(c_1^2+c_2^2-2c_1\cdot c_2\right) = \beta_2\left[(c_1\cdot c_2)+c_2^2\right]\left[c_1^2+c_2^2-2(c_1\cdot c_2)\right] = \beta_2\left[(c_1^2+c_2^2)\cancel{(c_1\cdot c_2)}^{0}-2\overbrace{(c_1\cdot c_2)^2}^{=\frac{1}{d}c_1^2c_2^2}+c_1^2c_2^2+c_2^4-2c_2^2\cancel{(c_1\cdot c_2)}^{0}\right]$$

$$c_{2i}\,2(c_{1j}+c_{2j})\frac{\beta_2}{d+2}\left(2g_ig_j+g^2\delta_{ij}\right) = \frac{2\beta_2}{d+2}\left[\left(c_{2i}c_{1j}+c_{2i}c_{2j}\right)2g_ig_j+g^2\delta_{ij}\left(c_{2i}c_{1j}+c_{2i}c_{2j}\right)\right]$$

$$= \frac{2\beta_2}{d+2}\left[\left(c_{2i}c_{1j}+c_{2i}c_{2j}\right)2(c_{1i}-c_{2i})(c_{1j}-c_{2j})+g^2\left[(c_1\cdot c_2)+c_2^2\right]\right]$$

$$= \frac{2\beta_2}{d+2}\left[2\left(c_{2i}c_{1j}+c_{2i}c_{2j}\right)\left(c_{1i}c_{1j}-c_{1i}c_{2j}-c_{2i}c_{1j}+c_{2i}c_{2j}\right)+\left(c_1^2+c_2^2-2c_1\cdot c_2\right)\left((c_1\cdot c_2)+c_2^2\right)\right]$$

$$= \frac{2\beta_2}{d+2}\Big[2\Big(\overset{(c_1\cdot c_2)c_1^2}{c_{2i}c_{1j}c_{1i}c_{1j}}-\overset{(c_1\cdot c_2)^2=\frac{1}{d}c_1^2c_2^2}{c_{2i}c_{1j}c_{1i}c_{2j}}-\overset{c_2^2c_1^2}{c_{2i}c_{1j}c_{2i}c_{1j}}+\overset{c_2^2(c_1\cdot c_2)}{c_{2i}c_{1j}c_{2i}c_{2j}}$$
$$+\underset{(c_1\cdot c_2)^2=\frac{1}{d}c_1^2c_2^2}{c_{2i}c_{2j}c_{1i}c_{1j}}-\underset{(c_1\cdot c_2)c_2^2}{c_{2i}c_{2j}c_{1i}c_{2j}}-\underset{(c_1\cdot c_2)c_2^2}{c_{2i}c_{2j}c_{2i}c_{1j}}+\underset{c_2^4}{c_{2i}c_{2j}c_{2i}c_{2j}}\Big)$$
$$+c_1^2c_2^2+c_2^4-2\cancel{(c_1\cdot c_2)}^{0}c_2^2+\cancel{(c_1\cdot c_2)}^{0}(c_1^2+c_2^2)-2\underbrace{(c_1\cdot c_2)^2}_{=\frac{1}{d}c_1^2c_2^2}\Big]$$

$$= \frac{2\beta_2}{d+2}\left[2\left\{\cancel{(c_1\cdot c_2)}^{0}c_1^2-c_2^2c_1^2-\cancel{(c_1\cdot c_2)}^{0}c_2^2+c_2^4\right\}+c_1^2c_2^2+c_2^4-\frac{2}{d}c_1^2c_2^2\right]$$

$$= \frac{2\beta_2}{d+2}\left[2c_2^4-2c_2^2c_1^2+c_1^2c_2^2+c_2^4-\frac{2}{d}c_1^2c_2^2\right]$$

$$= \frac{2\beta_2}{d+2}\left(3c_2^4-c_1^2c_2^2\frac{d+2}{d}\right)$$

$$-c_{2i}\beta_2\,g_j\left(c_1^2\delta_{ij}-c_2^2\delta_{ij}+2c_{1i}c_{1j}-2c_{2i}c_{2j}\right) = -\beta_2\,c_{2i}(c_{1j}-c_{2j})\left(c_1^2\delta_{ij}-c_2^2\delta_{ij}+2c_{1i}c_{1j}-2c_{2i}c_{2j}\right)$$

$$= -\beta_2\left(c_{1j}c_{2i}-c_{2i}c_{2j}\right)\left(c_1^2\delta_{ij}-c_2^2\delta_{ij}+2c_{1i}c_{1j}-2c_{2i}c_{2j}\right)$$

$\boxed{D}$

$$= -\beta_2\Big[c_{1j} c_{2i} c_1^2 \delta_{ij} - c_{1j} c_{2i} c_2^2 \delta_{ij} + c_{1j} c_{2i}\, 2 c_{1i} c_{1j} - c_{1j} c_{2i}\, 2 c_{2i} c_{2j} - c_{2i} c_{2j} c_1^2 \delta_{ij} + c_{2i} c_{2j} c_2^2 \delta_{ij} - c_{2i} c_{2j}\, 2 c_{1i} c_{1j} + c_{2i} c_{2j}\, 2 c_{2i} c_{2j}\Big]$$

$$= -\beta_2\Big[\cancel{(c_1\cdot c_2)}^{0} c_1^2 - \cancel{(c_1\cdot c_2)}^{0} c_2^2 + 2\cancel{(c_1\cdot c_2)}^{0} c_1^2 - 2\cancel{(c_1\cdot c_2)}^{0} c_2^2 - c_2^2 c_1^2 + c_2^4 - 2(c_1\cdot c_2)^2 + 2 c_2^4\Big]$$

$$= -\beta_2\Big[3 c_2^4 - c_1^2 c_2^2\left(1+\frac{2}{d}\right)\Big]$$

$$= -\beta_2\Big[3 c_2^4 - c_1^2 c_2^2\, \frac{d+2}{d}\Big]$$

Ainsi:

$$\int_{\mathbb{R}^d} dv\, S_i(v) L_c[M S_i] = -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4\pi^d} \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2}\left(c_2^2 - \frac{d+2}{2}\right)\beta_2\Big[\cancel{c_2^2}^{\,c_2^4 + c_1^2 c_2^2 \frac{d-2}{d}} + \left(3c_2^4 - c_1^2 c_2^2 \frac{d+2}{2}\right)\left(-1 + \frac{2}{d+2}\right)\Big]$$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4\pi^d}\beta_2 \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2}\left(c_2^2 - \frac{d+2}{2}\right)\Big[\cancel{c_2^2}^{\,c_2^4 + c_1^2 c_2^2 \frac{d-2}{d}} - \frac{d}{d+2}\left(3c_2^4 - c_1^2 c_2^2 \frac{d+2}{2}\right)\Big]$$ ⊛ cf. $\boxed{D2}$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4\pi^d}\beta_2 \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2}\Big[c_2^4 - \frac{d}{d+2}\left(3c_2^6 - c_1^2 c_2^4 \frac{d+2}{2}\right) - \frac{d+2}{2} c_2^2 + \frac{d}{2}\left(3c_2^4 - c_1^2 c_2^2 \frac{d+2}{2}\right)\Big]$$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4\pi^d}\beta_2 \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2}\Big[c_2^2\left(-\frac{d+2}{2} - \frac{d}{2}\frac{d+2}{2} c_1^2\right) + c_2^4\left(1 + \frac{3d}{2} + \frac{d}{d+2}\frac{d+2}{2} c_1^2\right) - c_2^6 \frac{3d}{d+2}\Big]$$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4\pi^d}\beta_2\, \pi^{d/2} \int_{\mathbb{R}^d} dc_1\, e^{-c_1^2}\Big[-\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)} \frac{d+2}{2}\left(1 + \frac{d}{2} c_1^2\right) + \frac{\Gamma(\frac{d+4}{2})}{\Gamma(d/2)}\left(1 + \frac{3d}{2} + \frac{d}{2} c_1^2\right) - \frac{\Gamma(\frac{d+6}{2})}{\Gamma(d/2)} \frac{3d}{d+2}\Big]$$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4}\beta_2 \Big[-\frac{d}{2}\frac{d+2}{2}\left(1 + \frac{d}{2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}\right) + \frac{d+2}{2}\frac{d}{2}\left(1 + \frac{3d}{2} + \frac{d}{2}\frac{\Gamma(\frac{d+2}{2})}{\Gamma(d/2)}\right) - \frac{d+4}{2}\frac{d+2}{2}\frac{d}{2}\frac{3d}{d+2}\Big]$$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4}\beta_2 \frac{d+2}{2}\frac{d}{2}\Big[\cancel{-1} - \cancel{\frac{d}{2}\frac{d}{2}} + \cancel{1} + \frac{3d}{2} + \cancel{\frac{d}{2}\frac{d}{2}} - \frac{d+4}{2}\frac{3d}{d+2}\Big]$$

$$= -\sigma^{d-1} \frac{\phi V_T^7}{S_d} \frac{m^2 n^2}{4}\beta_2 \frac{d+2}{2}\frac{d}{2}\frac{3d}{2}\Big[1 - \frac{d+4}{d+2}\Big]$$

Ainsi:

$$V_K^{*c} = \frac{2m\beta^3}{d(d+2)\, n V_0}\int_{\mathbb{R}^d} dv\, S_i(v) L_c[M S_i]$$

$$= \frac{\cancel{2}\cancel{m}\cancel{\beta^3}}{\cancel{d}\cancel{(d+2)}\cancel{n}} \cancel{\frac{\beta}{n}} \cancel{\frac{d+2}{8}} \cancel{\frac{\Gamma(d/2)}{\pi^{(d+1)/2}}} \sqrt{\cancel{\frac{m}{\beta}}} \cancel{\frac{1}{\sigma^{d-1}}}\cancel{\sigma^{d-1}} \cancel{\frac{K}{12}} \cancel{\frac{\pi^{(d+1)/2}}{\Gamma(d/2)}} \sqrt{\cancel{\frac{2}{\beta m}}} \left(\cancel{\frac{2}{\beta m}}\right)^3 \frac{\Gamma(d/2)}{2\pi^{d/2}} \frac{\cancel{m^2 n^2}}{4}\, 2\pi^{(d-1)/2} \frac{\Gamma(\frac{1+2}{2})}{\Gamma(\frac{d+2}{2})} \frac{d+2}{\cancel{2}}\frac{\cancel{d}}{\cancel{2}}\frac{3d}{\cancel{2}}\Big[\frac{d+4}{d+2} - \frac{d+2}{d+2}\Big]$$

$$= \frac{\cancel{\Gamma(d/2)}}{\cancel{2\pi^{d/2}}} \frac{1}{2} \cancel{\pi^{(d-1)/2}} \cancel{\frac{1}{2}}\cancel{\sqrt{\pi}} \frac{1}{\cancel{\frac{d}{2}}\cancel{\Gamma(d/2)}} \cancel{(d+2)}\, 3\cancel{d}\, \cancel{\frac{2}{d+2}}$$

$$\boxed{V_K^{*c} = \frac{3}{2}}$$

Conclusion:

$$V_K^{*c} = p\,\frac{d+2}{2} + \frac{3}{2}(1-p) \Rightarrow \boxed{K^* = \frac{d-1}{d}\frac{1}{V_K^*} = \frac{d-1}{d}\frac{2}{3}\frac{1}{p\,\frac{d+2}{3} + (1-p)}}$$

Pour avoir le résultat correct, on doit avoir :

$$K^* = \frac{1}{p\,\frac{d(d+2)}{2(d-1)} + (1-p)\,\underbrace{\frac{d}{d-1}\frac{3}{2}}_{=1}} \Rightarrow V_K^{*c} = \frac{d-1}{d}$$

$\boxed{D2}$

$$\int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} \left(c_2^2 - \frac{d+2}{2}\right)\left[c_2^4 + \frac{d-2}{d} c_1^2 c_2^2 - \frac{3d}{d+2} c_2^4 + c_1^2 c_2^2 \frac{d}{\cancel{d+2}} \cancel{\frac{d+2}{2}} \right]$$

$$= \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} \left(c_2^2 - \frac{d+2}{2}\right)\left[c_2^4 \left(\frac{d+2}{d+2} - \frac{3d}{d+2}\right) + c_1^2 c_2^2 \left(\frac{2(d-2)}{2d} + \frac{d^2}{2d}\right)\right]$$

$$= \int_{\mathbb{R}^{2d}} dc_1 dc_2\, e^{-c_1^2} e^{-c_2^2} \left[c_2^6 \frac{2-2d}{d+2} + c_1^2 c_2^4 \frac{d^2+2(d-2)}{2d} - c_2^4 \frac{d+2}{2} \frac{2-2d}{d+2} - c_1^2 c_2^2 \frac{d+2}{2} \frac{d^2+2(d-2)}{2d}\right]$$

$$= \pi^d \left[-2 \frac{d-1}{d+2} \frac{d+4}{2} \frac{d+2}{2} \frac{d}{2} + \frac{d}{2} \frac{d+2}{2} \frac{d}{2} \frac{d^2+2(d-2)}{2d} + 2 \frac{d+2}{2} \frac{d-1}{d+2} \frac{d+2}{2} \frac{d}{2} - \frac{d}{2} \frac{d}{2} \frac{d+2}{2} \frac{d^2+2(d-2)}{2d} \right]$$

$$= \pi^d \frac{d}{2} \frac{d+2}{2} \left[\frac{d-1}{d+2} \left(-\cancel{d} - 4 + \cancel{d} + 2 \right) + \frac{d^2+2(d-2)}{2d} \frac{d}{2} (1 - 1) \right]$$

$$= \pi^d \frac{d+2}{2} \frac{d}{2} \frac{d-1}{d+2} (-2)$$

$$= -\pi^d \frac{d+2}{2} \frac{d}{2} \frac{2(d-1)}{d+2}$$

Ainsi en reprenant les calculs de $\boxed{D}$, on a :

$$V_K^{*c} = \left(\cancel{\frac{3}{2}} \frac{\cancel{2}}{\cancel{3}d} \cancel{\frac{d+2}{2}} \right) \frac{\cancel{2}(d-1)}{\cancel{d+2}}$$

$$\boxed{V_K^{*} = \frac{d-1}{d}}$$

$$\Longrightarrow \boxed{K^* = \frac{1}{p \frac{d(d+2)}{2(d-1)} + (1-p)}}$$

<u>Ok</u> !